

Isogeny classes of Hilbert-Blumenthal abelian varieties over finite fields

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This paper gives an explicit formula for the size of the isogeny class of a Hilbert-Blumenthal abelian variety over a finite field. More precisely, let \mathcal{O}_L be the ring of integers in a totally real field dimension g over \mathbb{Q} , let N_0 and N be relatively prime square-free integers, and let k be a finite field of characteristic relatively prime to both N_0N and $\text{disc}(L, \mathbb{Q})$. Finally, let $(X/k, \iota, \alpha)$ be a g -dimensional abelian variety over k equipped with an action by \mathcal{O}_L and a $\Gamma_0(N_0, N)$ -level structure. Using work of Kottwitz, we express the number of $(X'/k, \iota', \alpha')$ which are isogenous to (X, ι, α) as a product of local orbital integrals on $\mathbf{GL}(2)$; then, using work of Arthur-Clozel and the affine Bruhat decomposition we evaluate all the relevant orbital integrals, thereby finding the cardinality of the isogeny class.

Key Words: base change, orbital integrals, abelian varieties, finite fields

In this paper we address the following question: Given an elliptic curve over a finite field, how many others are isogenous to it? While this is the question we address, we actually answer a question both more precise and more general. Let k be a finite field of characteristic p . Let L be a totally real field of dimension g , unramified at p , with ring of integers \mathcal{O}_L . For natural numbers N_0 and N relatively prime to p we may consider the Hilbert-Blumenthal variety \mathcal{M} , which is a fine moduli space for g -dimensional abelian varieties equipped with an action by \mathcal{O}_L and a $\Gamma_0(N_0, N)$ level structure. Two points in this space are called isogenous if there is a finite map between the associated abelian varieties which is compatible with their given \mathcal{O}_L -actions and level structures. Our initial goal is to measure the size of the isogeny class of a point defined over k ;

our subsequent goal is to describe how this cardinality is affected by changing base to finite extensions of k . These goals are realized in 3.1 and 4.1, respectively.

Our main result is the following.

THEOREM 3.1 *Suppose $(X, \iota, \alpha) \in \mathcal{M}(k)$ has no supersingular part. Thus, X is a g -dimensional abelian variety over a field with p^r elements, equipped with an action ι by the integers of a totally real field L and a $\Gamma_0(N_0, N)$ -level structure α , where p , N_0 , N and $\text{disc}(L, \mathbb{Q})$ are pairwise relatively prime and N_0N is square-free. Then the cardinality $|\mathcal{Y}_{(X, \iota, \alpha)}(k)|$ of the isogeny class of (X, ι, α) is*

$$p^{\frac{r}{2}g} h(M) \frac{[\mathbf{G}(\widehat{\mathbb{Z}}^p) : \Gamma(N)]}{\text{Norm}(N\mathcal{O}_L)} 2^{e(L, N_0)} C(\gamma, N_0, N) \prod_{w|\infty} \Delta_w(\gamma) \prod_{\pi|p} \binom{g_\pi}{a_\pi}$$

where:

- $\mathbf{G} \cong \text{res}_{\mathcal{O}_L, \mathbb{Z}} \mathbf{GL}_2$, and $\gamma \in \mathbf{G}(\widehat{\mathbb{Z}}_{\text{fin}}^p)$ is associated to X as in section 1;
- $h(M)$ is the class number of the ring $M = \text{End}_{\mathcal{O}_L}(X) \otimes \mathbb{Q}$;
- $e(L, N_0)$ is the number of primes λ dividing N_0 ;
- $\Delta_w(\gamma)$ is the norm of the discriminant of γ_w as in 2.9;
- $\{a_\pi\}_{\pi|p}$ determine the isogeny class of $X[p^\infty]$ as in 3.2, and $\binom{g_\pi}{a_\pi}$ is the binomial coefficient;
- $C(\gamma, N_0, N) = \prod_{l \neq p} \prod_{\lambda|l} C_\lambda(\gamma, N_0, N)$, where $\|\lambda\|$ denotes the cardinality of the residue field at λ , S is the set of all finite places λ of L such that $\text{ord}_\lambda(\text{disc}(M, L)) \equiv 1 \pmod{2}$, and

$$C_\lambda(\gamma, N_0, N) = \begin{cases} 1 & \gamma_\lambda \text{ split in } \mathbf{GL}_2(L_\lambda) \\ \frac{(1+\|\lambda\|)|2|_\lambda - 2\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N_0N \in \mathcal{O}_{L_\lambda}^\times \\ \frac{2|2|_\lambda \|\lambda\|^{1/2} - \Delta_\lambda(\gamma)(1+\|\lambda\|)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N_0 \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{(1+\|\lambda\|)(|2|_\lambda - \Delta_\lambda(\gamma))}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N_0 \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda \|\lambda\|^{1/2} - 2\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N_0N \in \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda(1+\|\lambda\|) - 2\|\lambda\|\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda(\|\lambda\|^{1/2} + \|\lambda\|^{3/2}) - 2\|\lambda\|\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N \notin \mathcal{O}_{L_\lambda}^\times \end{cases}$$

Our method owes everything to Kottwitz's careful description of the points on a Shimura variety of PEL type over a finite field. Following [K], we enumerate the elements of an isogeny class by means of certain orbital integrals on an algebraic group; then, using the fundamental lemma for cyclic base change on \mathbf{GL}_2 and the affine Bruhat decomposition for PGL_2 ,

we evaluate these orbital integrals and find the cardinality of the isogeny class.

The body of this paper starts with the introduction of notation and the recollection of constructions concerning Hilbert-Blumenthal moduli spaces; there is nothing new in this section. It concludes with a result describing the size of an isogeny class as an integral on $G = \text{res}_{\mathcal{O}_L, \mathbb{Z}} \mathbf{GL}_2$. The second section is a detailed study of certain (twisted) orbital integrals on G and \mathbf{GL}_2 . The third section applies the methods of the second section to the objects of the first, in order to produce an explicit formula for the size of the isogeny class of an \mathcal{O}_L -abelian variety over a finite field. The next section addresses the asymptotic issues described above. We close with an appendix which works out two concrete examples.

In the interest of efficiency we treat the Hilbert-Blumenthal case directly. With the exception of a new result on base change (2.4), however, all essential issues arise when $L = \mathbb{Q}$, that is, in counting elliptic curves. The reader looking for results on elliptic curves is encouraged to take L to be \mathbb{Q} ; read $X(N)$ for \mathcal{M} ; and compare the results with those in [S].

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1. MODULI SPACES

Fix a finite field $k \cong \mathbb{F}_{p^r}$. Let L be a totally real field of dimension $[L : \mathbb{Q}] = g$, unramified at p , such that all residue degrees g_π of L at p divide r . Let N_0 and N be square-free natural numbers such that $N \geq 3$ and N_0, N, p and $\text{disc}(L, \mathbb{Q})$ are pairwise relatively prime. This data determines a certain representable Hilbert-Blumenthal moduli problem; we refer to [R] and [D-P] for standard results concerning such.

DEFINITION 1.1. Let \mathcal{M} be the fine moduli scheme parameterizing isomorphism classes of data $(X/S, \iota, \alpha)$ where: $X \rightarrow S \rightarrow \mathbb{Z}[\frac{1}{N \text{disc}_{K, \mathbb{Q}}}]$ is an abelian scheme of dimension g ; $\mathcal{O}_L \xrightarrow{\iota} \text{End}_S(X)$ is an action sending $1 \in \mathcal{O}_L$ to id_X , so that $\text{Lie}(X/S)$ is a locally free rank-one $\mathcal{O}_L \otimes \mathcal{O}_S$ -module; and α is a $\Gamma_0(N_0, N)$ -level structure on X . (By this, we mean that α encodes both a full level N structure and a $\Gamma_0(N_0)$ level structure; see [P, 2.2].)

DEFINITION 1.2. For such an object $(X/k, \iota, \alpha)$ and a k -scheme S , let $\mathcal{Y}_{(X, \iota, \alpha)}(S)$ denote the set of all isomorphism classes $(X'/S, \iota', \alpha')$ which are isogenous to $(X \times S, \iota \times S, \alpha \times S)$.

We will use the homology of X and proposition 1.1 below to apprehend $\mathcal{Y}_{(X,\iota,\alpha)}(k)$. Let \mathcal{O}_E be the ring of Witt vectors of k , let $E = \text{Frac } \mathcal{O}_E$, and let σ be the lift to \mathcal{O}_E of the Frobenius automorphism of k . Let π_X be the map induced by Frobenius on prime-to- p étale homology $H_1^{\text{ét}}(X, \widehat{\mathbb{Z}}^p)$, and let Φ be the Frobenius of the crystalline homology $H_1^{\text{cris}}(X, \mathcal{O}_E)$. Via the action of \mathcal{O}_L these homology groups are free, rank-two $\mathcal{O}_L \otimes \widehat{\mathbb{Z}}^p$ and $\mathcal{O}_L \otimes \mathcal{O}_E$ -modules, respectively. We analyze X and other elements of $\mathcal{Y}_{(X,\iota,\alpha)}(k)$ by comparing their integral homology with a reference module.

DEFINITION 1.3. Let V be a free, rank two \mathcal{O}_L -module. Let G/\mathbb{Z} be the algebraic group $\mathbf{GL}_{\mathcal{O}_L}(V)$. Thus, $G \cong \text{res}_{\mathcal{O}_L, \mathbb{Z}} \mathbf{GL}_2$, and

$$G(R) \cong \mathbf{GL}_2(\mathcal{O}_L \otimes R),$$

for any ring R .

Fix an isomorphism of $\mathcal{O}_L \otimes (\widehat{\mathbb{Z}}^p \times \mathcal{O}_E)$ modules

$$H_1^{\text{ét}}(X, \widehat{\mathbb{Z}}^p) \times H_1^{\text{cris}}(X, \mathcal{O}_E) \xrightarrow{\sim} (V \otimes \widehat{\mathbb{Z}}^p) \times (V \otimes \mathcal{O}_E).$$

Under this identification, π_X gives rise to an element $\gamma \in G(\mathbb{A}_{\text{fin}}^p)$ and to Φ corresponds some $\delta \in G(E)$. The conjugacy class of γ and the σ -conjugacy class of σ are independent of the choice of isomorphism.

Let M be the \mathbb{Q} -algebra of \mathcal{O}_L -linear endomorphisms of (X, ι) ; thus $M = \text{End}_{\mathcal{O}_L}(X) \otimes \mathbb{Q}$. Let T be the associated group scheme, so that $T(R) = (M \otimes R)^\times$. By the work of Tate on endomorphisms of abelian varieties (cf. [T2], or [K] 10.7 and 10.8), $T(\mathbb{Q}_l)$ is the centralizer of γ in $G(\mathbb{Q}_l)$, and $T(\mathbb{Q}_p)$ is the σ -twisted centralizer of δ in $G(E)$.

Let $K(N_0, N)^p$ be the intersection of

$$\Gamma_0(N_0) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\widehat{\mathbb{Z}}^p) \mid N \text{ divides } c \right\}$$

and

$$\Gamma(N) := \ker(G(\widehat{\mathbb{Z}}^p) \rightarrow \prod_{l|N} G(\mathbb{Z}_l) \rightarrow \prod_{l|N} G(\mathbb{Z}_l/l\mathbb{Z}_l)).$$

Let $K_p = G(\mathcal{O}_E)$, and let $K(N_0, N) = K(N_0, N)^p \times K_p$. Thus, since N_0 and N are square-free and relatively prime, $K(N_0, N)_l$ is either the maximal compact subgroup $G(\mathbb{Z}_l)$, an Iwahori subgroup contained in $G(\mathbb{Z}_l)$, or the kernel of the reduction map $G(\mathbb{Z}_l) \rightarrow G(\mathbb{Z}_l/l\mathbb{Z}_l)$.

DEFINITION 1.4. Equip $T(\mathbb{Q})$ with the counting measure, $T(\mathbb{A}_{\text{fin}}^p)$ with the Haar measure giving $T(\widehat{\mathbb{Z}}^p)$ measure 1, $G(\mathbb{A}_{\text{fin}}^p)$ with the Haar measure

giving $K(N_0, N)^p$ measure 1. For any function $f \in C_c^\infty(G(\mathbb{A}_{\text{fin}}^p))$, denote the orbital integral of f at γ by

$$\Phi_{G(\mathbb{A}_{\text{fin}}^p)}(\gamma, f) := \int_{T(\mathbb{A}_{\text{fin}}^p) \backslash G(\mathbb{A}_{\text{fin}}^p)} f(x^{-1}\gamma x) d\bar{x},$$

where $d\bar{x}$ denotes the quotient measure. Similarly, equip $G(E)$ with the unique Haar measure giving K_p measure 1, and $T(\mathbb{Q}_p)$ with the measure giving $T(\mathbb{Z}_p)$ measure 1. For a Schwartz function $\phi \in C_c^\infty(G(E))$, denote the twisted orbital integral of ϕ at δ by

$$\Phi_{G(E)}^\sigma(\delta, \phi) := \int_{T(\mathbb{Q}_p) \backslash G(E)} \phi(x^{-1}\delta x^\sigma) d\bar{x},$$

where $d\bar{x}$ denotes the quotient measure.

PROPOSITION 1.1. *For any $(X, \iota, \alpha) \in \mathcal{M}(k)$,*

$$|\mathcal{Y}_{(X, \iota, \alpha)}(k)| = \text{vol}(T(\mathbb{Q}) \backslash T(\mathbb{A}_{\text{fin}})) \Phi_{G(\mathbb{A}_{\text{fin}}^p)}(\gamma, f_{\text{fin}}^p) \Phi_{G(E)}^\sigma(\delta, \phi),$$

where f_{fin}^p is the characteristic function of $K(N_0, N)^p$ and ϕ is the characteristic function of the bi- $G(\mathcal{O}_E)$ coset of $\text{diag}(1, p)$.

Proof: The statement and proof are both essentially due to Kottwitz (see section 16 of [K]). Speaking very roughly, an element $x \times y \in G(\mathbb{A}_{\text{fin}}^p) \times G(E)$ determines a new lattice in the rational homology of X ; we seek to count those associated to the integral homology of some element of $\mathcal{Y}_{(X, \iota, \alpha)}(k)$. The twisted orbital integral insures that the crystalline homology is Frobenius-stable, and that the tangent space of the associated abelian variety is free over $\mathcal{O}_L \otimes k$. The orbital integral insures that the l -adic homology is also Frobenius stable, and that the abelian variety supports an appropriate level structure. In fact, we work with a slightly different moduli space than those of [K]. The distinction requires an investigation of \mathcal{O}_L -linear polarizations on Hilbert-Blumenthal abelian varieties. Since these notions never arise again in the present work, we refer the reader to the opening section of each of [R] and [D-P] for relevant background.

While we work with the group G defined above, the analogous group in [K] is the sub- \mathbb{Z} -group scheme $G' \subset G$ of elements with rational determinant. It is associated to a moduli scheme \mathcal{M}' which parameterizes data $(X, \iota, \alpha, \beta)$, where X , ι , and α are as above and β is a \mathbb{Q}^\times -class of \mathcal{O}_L -polarizations. Now, any Hilbert-Blumenthal abelian variety (X, ι) admits an \mathcal{O}_L -polarization ([R] 1.12). Moreover, any two such differ by an element

of L^\times ([R] 1.17). Thus, although we have omitted it from the definition, \mathcal{M} parameterizes data $(X, \iota, \alpha, \beta)$, where β is an L^\times -class of \mathcal{O}_L -polarizations. (Re)introducing β allows for a simple adaptation of Kottwitz’s proof. Replacing “polarization up to \mathbb{Q}^\times ” with “polarization up to L^\times ” on page 431 of [K] proves proposition 1.1. ■

The next section provides tools for evaluating the integrals which come up in the statement of 1.1. Actually, we will only compute $|\mathcal{Y}_{(X, \iota, \alpha)}(k)|$ for those X with no supersingular part (as in the beginning of section 3). Since X has no supersingular part, one knows that the commutant $(\text{End}_{\mathcal{O}_L}(X) \otimes \mathbb{Q})^\times$ lies in a torus. Because of this, and the usual semisimplicity of Frobenius [T], we will assume that γ and δ are regular semisimple.

2. LOCAL ORBITAL INTEGRALS

To evaluate the orbital integrals in proposition 1.1, we must evaluate a countably-infinite number of local orbital integrals. Each of these local orbital integrals is treated by one of the eight propositions this section.

In subsection 2.1 we express the twisted local orbital integrals on G from proposition 1.1 as split local orbital integrals on GL_2 ; the main result is proposition 2.1.

In subsection 2.2, we prove four results: proposition 2.2 evaluates the split local orbital integrals appearing in proposition 2.1; proposition 2.3 evaluates split local orbital integrals of the form $\Phi_{\text{GL}_2(F)}(t, f)$ when f is the characteristic function of an Iwahori subgroup \mathcal{I} ; proposition 2.4 evaluates split local orbital integrals when f is the characteristic function of a maximal compact subgroup K ; finally, proposition 2.5 evaluates split local orbital integrals when f is the characteristic function of K_+ (defined below).

In subsection 2.3 we prove three results: proposition 2.7 evaluates regular elliptic local orbital integrals of the form $\Phi_{\text{GL}_2(F)}(\gamma, f)$ when f is the characteristic function \mathcal{I} ; proposition 2.8 evaluates regular elliptic local orbital integrals when f is the characteristic function of K ; and proposition 2.9 evaluates regular elliptic local orbital integrals when f is the characteristic function of K_+ .

2.1. Twisted Orbital integrals and the Fundamental Lemma

DEFINITION 2.1. In subsection 2.1 the local field E is an unramified degree- r extension of \mathbb{Q}_p and σ is a generator for the cyclic Galois group $\text{Gal}(E/\mathbb{Q}_p)$. We equip $G(E)$ with the unique Haar measure giving K_p measure 1 and equip a maximal torus $T(\mathbb{Q}_p)$ with the unique Haar measure giving $T(\mathbb{Z}_p)$ measure 1. For a Schwartz function $\phi \in C_c^\infty(G(E))$, denote

the twisted orbital integral of ϕ at $\delta \in T(\mathbb{Q}_p)$ by

$$\Phi_{\mathbf{G}(E)}^\sigma(\delta, \phi) := \int_{T(\mathbb{Q}_p) \backslash \mathbf{G}(E)} \phi(x^{-1} \delta x^\sigma) d\bar{x},$$

where $d\bar{x}$ denotes the quotient measure on $T(\mathbb{Q}_p) \backslash \mathbf{G}(E)$ determined by these choices.

In this section we use the Fundamental Lemma for base change on GL_2 as in [A-C] to study the twisted orbital integral on $\mathbf{G}(E)$ appearing in proposition 1.1. Recall that E is an unramified degree- r extension of \mathbb{Q}_p and that σ is a generator for the cyclic Galois group $\mathrm{Gal}(E/\mathbb{Q}_p)$. Recall also that L is a totally real degree- g number field.

Fix a prime ideal π of L dividing p , and let L_π be the completion of L at π . Recall that g_π is the degree $[L_\pi : \mathbb{Q}_p]$. In this section, we will write τ for σ^{g_π} . Let $\mathbf{G}_\pi = \mathrm{res}_{\mathcal{O}_{L_\pi, \mathbb{Z}_p}} \mathrm{GL}_2$, so $\mathbf{G}_\pi(E) = \mathrm{GL}_2(L_\pi \otimes_{\mathbb{Z}_p} E)$.

DEFINITION 2.2. Choose an embedding ρ_π of L_π into E with image F , where F is the subfield of E fixed by τ . Define $\rho_i := \sigma^{i-1} \circ \rho_\pi$ for each $i = 1, \dots, g_\pi$. Define $\rho : L_\pi \otimes_{\mathbb{Z}_p} E \rightarrow E \times \cdots \times E$ by $\rho(a \otimes b) = (\rho_1(a)b, \dots, \rho_{g_\pi}(a)b)$. We also let ρ denote the induced map $\mathbf{G}_\pi(E) \rightarrow \mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E)$, and write $\rho(x) = (x_1, \dots, x_{g_\pi})$.

LEMMA 2.1. *The induced map $\rho : \mathbf{G}_\pi(E) \rightarrow \mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E)$ is an isomorphism of algebraic groups over F . The natural action of $\mathrm{Gal}(E/\mathbb{Q}_p)$ on $\mathbf{G}_\pi(E)$ corresponds with the action of $\mathrm{Gal}(E/\mathbb{Q}_p)$ on $\mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E)$ defined by*

$$(x_1, \dots, x_{g_\pi})^\sigma = (x_{g_\pi}^\sigma, x_1^\sigma, \dots, x_{g_\pi-1}^\sigma).$$

Proof: By the natural action we mean that $\mathrm{Gal}(E/\mathbb{Q}_p)$ acts on $L_\pi \otimes_{\mathbb{Q}_p} E$ via the second component; that is, $\sigma(a \otimes b) = a \otimes \sigma(b)$. Let $z = a \otimes b$ and write $\rho(z) = (z_1, \dots, z_{g_\pi})$; thus $z_i = \rho_i(a)b$, for $i = 1, \dots, g_\pi$. Let $w = z^\sigma$ and let $\rho(w) = (w_1, \dots, w_{g_\pi})$. On one hand, for $i = 2, \dots, g_\pi$ we have

$$\begin{aligned} w_i &= \rho_i(a)\sigma(b) \\ &= \sigma^{i-1}(\rho_\pi(a))\sigma(b) \\ &= \sigma(\sigma^{i-2}(\rho_\pi(a))b) \\ &= \sigma(\rho_{i-1}(a)b) \\ &= z_{i-1}^\sigma. \end{aligned}$$

On the other hand, since $\rho_\pi(a)$ is fixed by $\tau = \sigma^{g_\pi}$, we have

$$\begin{aligned} w_1 &= \rho_1(a)\sigma(b) \\ &= \sigma^g(\rho_\pi(a))\sigma(b) \\ &= \sigma(\sigma^{g_\pi-1}(\rho_\pi(a))b) \\ &= \sigma(\rho_{g_\pi}(a)b) \\ &= z_{g_\pi}^\sigma. \end{aligned}$$

Thus,

$$\rho(\sigma(z)) = (z_{g_\pi}^\sigma, z_1^\sigma, \dots, z_{g_\pi-1}^\sigma).$$

This describes the action of $\text{Gal}(E/\mathbb{Q}_p)$ on $E \times \cdots \times E$ given by the isomorphism ρ defined above. The action of $\text{Gal}(E/\mathbb{Q}_p)$ on $\text{GL}_2(E) \times \cdots \times \text{GL}_2(E)$ now follows. \blacksquare

LEMMA 2.2. *The twisted diagonal embedding*

$$x_1 \mapsto (x_1, x_1^\sigma, x_1^{\sigma^2}, \dots, x_1^{\sigma^{g_\pi-1}})$$

of $\text{GL}_2(E)$ into $\text{GL}_2(E) \times \cdots \times \text{GL}_2(E)$ defines an isomorphism $\text{GL}_2(F) \rightarrow \text{G}_\pi(E)^{\text{Gal}(E/\mathbb{Q}_p)}$ of algebraic groups over F .

Proof: Since σ generates $\text{Gal}(E/\mathbb{Q}_p)$ it is clear that $x \in \text{G}_\pi(E)^{\text{Gal}(E/\mathbb{Q}_p)}$ if and only if $x^\sigma = x$. Now, let $\rho(x) = (x_1, \dots, x_{g_\pi})$. Then $\rho(x^\sigma) = (x_{g_\pi}^\sigma, x_1^\sigma, \dots, x_{g_\pi-1}^\sigma)$, so $x^\sigma = x$ if and only if $x_1 = x_{g_\pi}^\sigma$ and $x_i = x_{i-1}^\sigma$ for $i = 2, \dots, g_\pi$. Thus, in this case, $x_1 = x_1^{\sigma^{g_\pi}}$ and $x_i = x_1^{\sigma^{i-1}}$ for $i = 2, \dots, g_\pi$. Recall that $\tau := \sigma^{g_\pi}$ and notice that the condition $x_1^\tau = x_1$ is equivalent to the assertion that $x_1 \in \text{GL}_2(F)$. This proves the lemma. \blacksquare

DEFINITION 2.3. Define $N_\pi : \text{G}_\pi(E) \rightarrow \text{GL}_2(E)$ by

$$N_\pi(x) = x_{g_\pi} x_{g_\pi-1}^\sigma x_{g_\pi-2}^{\sigma^2} \cdots x_2^{\sigma^{g_\pi-2}} x_1^{\sigma^{g_\pi-1}},$$

where $x_i = \rho_i(x)$ as in definition 2.2.

LEMMA 2.3. *If $\delta \in \text{G}_\pi(E)$ and $\xi = N_\pi(\delta)$, then the τ -twisted centralizer $C_{\text{GL}_2(E)}^\tau(\xi)$ of ξ in $\text{GL}_2(E)$ is isomorphic over F to the σ -twisted centralizer $C_{\text{G}_\pi(E)}^\sigma(\delta)$ of δ in $\text{G}_\pi(E)$.*

Proof: By definition, $C_{G_\pi(E)}^\sigma(\delta) := \{x \in G_\pi(E) \mid x^{-1}\delta x^\sigma = \delta\}$. Writing $\rho(x) = (x_1, \dots, x_{g_\pi})$ as in lemma 2.2,

$$\rho(x^{-1}\delta x^\sigma) = (x_1^{-1}\delta_1 x_{g_\pi}^\sigma, x_2^{-1}\delta_2 x_1^\sigma, \dots, x_{g_\pi}^{-1}\delta_{g_\pi} x_{g_\pi-1}^\sigma).$$

Since ρ is a bijection, $x^{-1}\delta x^\sigma = \delta$ if and only if $\rho(x^{-1}\delta x^\sigma) = \rho(\delta)$. Thus, $x_1^{-1}\delta_1 x_{g_\pi}^\sigma = \delta_1$ and $x_i^{-1}\delta_i x_{i-1}^\sigma = \delta_i$ for all $i = 2, \dots, g_\pi$. It follows that $x_{g_\pi}^{-1}\xi x_{g_\pi}^\tau = x_{g_\pi}$ and $x_i = (\delta_i \delta_{i-1}^\sigma \cdots \delta_1^{\sigma^{i-1}}) x_g^{\sigma^i} (\delta_i \delta_{i-1}^\sigma \cdots \delta_1^{\sigma^{i-1}})^{-1}$ for all $i = 1, \dots, g_\pi - 1$. Now define $f : \mathrm{GL}_2(E) \rightarrow \mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E)$ by $f(t) = (t_1, \dots, t_{g_\pi})$ where

$$\begin{aligned} t_1 &= \delta_1 t^\sigma \delta_1^{-1} \\ t_2 &= (\delta_2 \delta_1^\sigma) t^{\sigma^2} (\delta_2 \delta_1^\sigma)^{-1} \\ &\dots \\ t_{g_\pi-1} &= (\delta_{g_\pi-1} \delta_{g_\pi-2}^\sigma \cdots \delta_1^{\sigma^{g_\pi-2}}) t^{\sigma^{g_\pi-1}} (\delta_{g_\pi-1} \delta_{g_\pi-2}^\sigma \cdots \delta_1^{\sigma^{g_\pi-2}})^{-1} \\ t_{g_\pi} &= \xi t^\tau \xi^{-1}. \end{aligned}$$

Then $\rho^{-1} \circ f$ restricts to an isomorphism from $C_{\mathrm{GL}_2(E)}^\tau(\xi)$ to $C_{G_\pi(E)}^\sigma(\delta)$. Since ρ and f are each morphisms of algebraic groups defined over F , the lemma is proved. \blacksquare

DEFINITION 2.4. Let $\mathcal{H}_\pi(E)$ denote the convolution algebra of bi- $G_\pi(\mathcal{O}_E)$ functions on $G_\pi(E)$ with compact support modulo the centre of $G_\pi(E)$. Also, let \mathcal{H}_E denote the convolution algebra of bi- $\mathrm{GL}_2(\mathcal{O}_E)$ functions on $\mathrm{GL}_2(E)$ with compact support modulo the centre of $\mathrm{GL}_2(E)$.

LEMMA 2.4. Suppose ϕ is an element of $\mathcal{H}_\pi(E)$ such that

$$\phi \circ \rho^{-1} = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_{g_\pi},$$

where $\phi_i \in \mathcal{H}_E$. Let δ be any element of $G_\pi(E)$ and set $\xi = N_\pi(\delta)$, where N_π is given by definition 2.3. Then

$$\Phi_{G_\pi(E)}^\sigma(\delta, \phi) = \Phi_{\mathrm{GL}_2(E)}^\tau(\xi, \psi),$$

where

$$\psi := \phi_{g_\pi} * \phi_{g_\pi-1}^\sigma * \cdots * \phi_1^{\sigma^{g_\pi-1}}.$$

Here we define ϕ_i^α by $\phi_i^\alpha(y_i^\alpha) = \phi(y_i)$ for all $\alpha \in \mathrm{Gal}(E/\mathbb{Q}_p)$ and all $y_i \in \mathrm{GL}_2(E)$.

Proof: To simplify notation slightly we write g for g_π in this proof only. By definition 2.8,

$$\Phi_{G_\pi(E)}^\sigma(\delta, \phi) := \int_{C_{G_\pi(E)}^\sigma(\delta) \backslash G_\pi(E)} \phi(x^{-1} \delta x^\sigma) d\bar{x}.$$

Using the isomorphism ρ from definition 2.2 and the isomorphism f from the proof of lemma 2.3 we re-write the σ -twisted orbital integral on $G_\pi(E)$ as an orbital integral on $\mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E)$; specifically, if $\rho(\delta) = (\delta_1, \dots, \delta_g)$ then $\Phi_{G_\pi(E)}^\sigma(\delta, \phi)$ is equal to

$$f(C_{\mathrm{GL}_2(E)}^\tau(\xi)) \backslash \mathrm{GL}_2(E) \times \cdots \times \mathrm{GL}_2(E) \int \phi_1(x_1^{-1} \delta_1 x_1^\sigma) \cdots \phi_g(x_g^{-1} \delta_g x_{g-1}^\sigma) d\bar{x}.$$

Here we use the function f to push a Haar measure on $C_{\mathrm{GL}_2(E)}^\tau(\xi)$ to the measure dt on $C_{G_\pi(E)}^\sigma(\delta)$ and then write $d\bar{x}$ for the quotient measure $d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_g$. Suppressing the domain of integration for the moment we invoke Fubini's theorem and the definition of ϕ_i^α to write this integral as

$$\int \phi_g(x_g^{-1} \delta_g x_{g-1}^\sigma) \cdots \phi_1^{\sigma^{g-1}}((x_1^{\sigma^{g-1}})^{-1} \delta_1^{\sigma^{g-1}} x_g^\tau) d\bar{x}.$$

Adapting ideas from [A-C] we introduce the following change-of-variables defined over F :

$$\begin{aligned} y_g &= x_g \\ y_{g-1} &= (x_{g-1}^\sigma)^{-1} \delta_{g-1}^{-1} \xi x_g^\tau \\ y_{g-2} &= (x_{g-2}^{\sigma^2})^{-1} (\delta_{g-1}^\sigma)^{-1} \delta_g^{-1} \xi x_g^\tau \\ &\dots \\ y_2 &= (x_2^{\sigma^{g-2}})^{-1} (\delta_3^{\sigma^{g-3}})^{-1} \cdots \delta_g^{-1} \xi x_g^\tau \\ y_1 &= (x_1^{\sigma^{g-1}})^{-1} (\delta_2^{\sigma^{g-2}})^{-1} \cdots \delta_g^{-1} \xi x_g^\tau. \end{aligned}$$

Using these new variables we find

$$\begin{aligned} \phi_g(x_g^{-1} \delta_g x_{g-1}^\sigma) &= \phi_g(y_g^{-1} \xi y_g^\tau y_{g-1}^{-1}) \\ \phi_{g-1}^\sigma((x_{g-1}^\sigma)^{-1} \delta_{g-1}^\sigma x_{g-2}^{\sigma^2}) &= \phi_{g-1}^\sigma(y_{g-1} y_{g-2}^{-1}) \\ \phi_{g-2}^{\sigma^2}((x_{g-2}^{\sigma^2})^{-1} \delta_{g-2}^{\sigma^2} x_{g-3}^{\sigma^3}) &= \phi_{g-2}^{\sigma^2}(y_{g-2} y_{g-3}^{-1}) \\ &\dots \\ \phi_2^{\sigma^{g-2}}((x_2^{\sigma^{g-2}})^{-1} \delta_1^{\sigma^{g-2}} x_1^{\sigma^{g-1}}) &= \phi_2^{\sigma^{g-2}}(y_2 y_1^{-1}) \end{aligned}$$

$$\phi_1^{\sigma^{g-1}}((x_1^{\sigma^{g-1}})^{-1} \delta_1^{\sigma^{g-1}} x_g^\tau) = \phi_1^{\sigma^{g-1}}(y_1).$$

Now, let $f(y_1, \dots, y_g)$ denote the function

$$\phi_g(y_g^{-1} \xi y_g^\tau y_{g-1}^{-1}) \phi_{g-1}^\sigma(y_{g-1} y_{g-2}^{-1}) \phi_{g-2}^{\sigma^2}(y_{g-2} y_{g-3}^{-1}) \cdots \phi_2^{\sigma^{g-2}}(y_2 y_1^{-1}) \phi_1^{\sigma^{g-1}}(y_1),$$

and write $d\bar{y}$ for the quotient measure $d\bar{y}_1 d\bar{y}_2 \cdots d\bar{y}_g$. Then

$$\begin{aligned} \Phi_{G_\pi(E)}^\sigma(\delta, \phi) &= \int f(y) d\bar{y} \\ &= \int_{C_{\mathrm{GL}_2(E)}^\tau(\xi) \backslash \mathrm{GL}_2(E)} (\phi_{g_\pi} * \phi_{g_\pi-1}^\sigma * \cdots * \phi_1^{\sigma^{g_\pi-1}})(y_g^{-1} \xi y_g^\tau) d\bar{y}_g \\ &= \int_{C_{\mathrm{GL}_2(E)}^\tau(\xi) \backslash \mathrm{GL}_2(E)} \psi(g^{-1} \xi g^\tau) d\bar{g} \\ &= \Phi_{\mathrm{GL}_2(E)}^\tau(\xi, \psi). \end{aligned}$$

This proves the lemma. \blacksquare

DEFINITION 2.5. For any p-adic field E , the Satake transform $\mathcal{S}f \in \mathbb{C}[T_1, T_2, T_1^{-1}, T_2^{-1}]$ of $f \in \mathcal{H}_E$ is

$$\mathcal{S}f(T_1, T_2) := \sum_{n_1, n_2 \in \mathbb{Z}} f_{n_1, n_2} T_1^{n_1} T_2^{n_2}$$

where

$$f_{n_1, n_2} := q_E^{(n_2 - n_1)/2} \int_{x \in F} f\left(\begin{pmatrix} \varpi_F^a & x \varpi_F^a \\ 0 & \varpi_F^b \end{pmatrix}\right) dx.$$

Here, q_E is the cardinality of the residue field of E and the measure dx on E has been normalized so that the measure of \mathcal{O}_E is 1.

Remark: Without proof we remark that \mathcal{S} is an isomorphism from the convolution \mathbb{C} -algebra \mathcal{H}_F to the \mathbb{C} -algebra $\mathbb{C}[T_1, T_2, T_1^{-1}, T_2^{-1}]^{S_2}$ of symmetric Laurent polynomials in two variables. The interested reader is referred to [Sa].

DEFINITION 2.6. Return now to E and F as defined at the beginning of this section. Let $b_{E/F} : \mathcal{H}_E \rightarrow \mathcal{H}_F$ denote the base-change homomorphism from $\mathrm{GL}_2(E)$ to $\mathrm{GL}_2(F)$, and let $N_{E/F} : \mathrm{GL}_2(E) \rightarrow \mathrm{GL}_2(E)$ be as in [L].

Remark: Observe that E is a cyclic extension of F of degree $s_\pi := r/g_\pi$. Thus, the base-change map $b_{E/F}$ is completely characterized by

$$\mathcal{S}_F \circ b_{E/F} \circ \mathcal{S}_E^{-1} : f(T_1, T_2) \mapsto f(T_1^{s_\pi}, T_2^{s_\pi}),$$

for all $f \in \mathbb{C}[T_1, T_2, T_1^{-1}, T_2^{-1}]^{S_2}$. The reader is referred to [L] for a proof of this statement. The function $N_{E/F} : \mathrm{GL}_2(E) \rightarrow \mathrm{GL}_2(E)$ is given by

$$N_{E/F}(\xi) = \xi \xi^\tau \xi^{\tau^2} \cdots \xi^{\tau^{s_\pi-1}}.$$

Finally we may recall the version of the Fundamental Lemma which lies at the heart of this paper. For a proof the reader is referred to [L] or [A-C].

LEMMA 2.5. *Let ψ be any element of \mathcal{H}_E and suppose that ξ is a regular semi-simple element of $\mathrm{GL}_2(E)$. Then*

$$\Phi_{\mathrm{GL}_2(E)}^\tau(\xi, \psi) = \Phi_{\mathrm{GL}_2(F)}(\gamma, f),$$

where $f = b_{E/F}(\psi)$ and where γ is any element of $\mathrm{GL}_2(F)$ which is $\mathrm{GL}_2(F)$ -conjugate to $N_{E/F}(\xi)$.

The remainder of this section is concerned with lemmas 2.4 and 2.5 as they apply to our calculation of isogeny classes of Hilbert-Blumenthal abelian varieties in section 3.

DEFINITION 2.7. Let ϕ_π denote the characteristic function of the bi- $\mathrm{G}_\pi(\mathcal{O}_E)$ -coset of $\mathrm{diag}(1 \otimes 1, p \otimes 1)$ and let $\delta_\pi \in \mathrm{G}_\pi(E)$ represent the σ -conjugacy class of the L_π -part of the crystalline homology of a Hilbert-Blumenthal abelian variety. (For this last notion, see the discussion immediately preceding 3.1; the isomorphism $L \otimes \mathbb{Q}_p \cong \bigoplus_{\pi|p} L_\pi$ induces a decomposition of the first homology of an \mathcal{O}_L -abelian variety.) Finally, let $\gamma_\pi = (N_{E/F} \circ N_\pi)(\delta_\pi)$ and note that $\gamma_\pi \in \mathrm{GL}_2(L_\pi)$.

PROPOSITION 2.1. *Let ϕ_π and δ_π be as in definition 2.7. Then*

$$\Phi_{\mathrm{G}_\pi(E)}^\sigma(\delta_\pi, \phi_\pi) = \Phi_{\mathrm{GL}_2(L_\pi)}(\gamma_\pi, f_\pi),$$

where the Satake transform of f_π is

$$\mathcal{S}f_\pi = (p^{r/2})^{g_\pi} (T_1^{s_\pi} + T_2^{s_\pi})^{g_\pi}.$$

Proof: From the definition of ϕ_π it follows that $\phi_\pi \circ \rho^{-1} = \phi_0 \otimes \cdots \otimes \phi_0$, where $\phi_0 \in \mathcal{H}_E$ is the characteristic function of the bi- $\mathrm{GL}_2(\mathcal{O}_E)$ -coset of $\mathrm{diag}(1, p)$. Observe that p is a uniformizer in E , so $\phi_0^\sigma = \phi_0$. By lemma 2.4,

$$\Phi_{\mathrm{GL}_2(E)}^\sigma(\delta_\pi, \phi_\pi) = \Phi_{\mathrm{GL}_2(E)}^\tau(\xi_\pi, \psi_\pi),$$

where $\xi_\pi = N_\pi(\delta_\pi)$ and where $\psi_\pi = \phi_0 * \cdots * \phi_0$. Notice that $\mathcal{S}\psi_\pi = (\mathcal{S}\phi_0)^{g_\pi}$. Using definition 2.5 we see that $\mathcal{S}\phi_0(T_1, T_2) = p^{r/2}(T_1 + T_2)$, so $\mathcal{S}\psi_\pi(T_1, T_2) = (p^{r/2})^{g_\pi}(T_1 + T_2)^{g_\pi}$. By lemma 2.5,

$$\Phi_{\mathrm{GL}_2(E)}^\tau(\xi_\pi, \psi_\pi) = \Phi_{\mathrm{GL}_2(L_\pi)}^\tau(\gamma_\pi, f_\pi),$$

where $f_\pi := b_{E/L_\pi}(\psi_\pi)$. Finally, observe that the Satake transform of f_π is $\mathcal{S}f_\pi = (p^{r/2})^{g_\pi}(T_1^{s_\pi} + T_2^{s_\pi})^{g_\pi}$. This completes the proof of proposition 2.1. \blacksquare

2.2. Split orbital integrals

DEFINITION 2.8. In subsections 2.2 and 2.3, F denotes an arbitrary characteristic zero local field with integers \mathcal{O}_F , units \mathcal{O}_F^\times , prime ideal \mathfrak{p}_F , uniformizer ϖ_F and residue field k_F . The norm $|\cdot|_F$ is determined by the condition $|\varpi_F|_F = q_F^{-1}$, where q_F is the cardinality of the residue field of F . Equip $\mathrm{GL}_2(F)$ with the unique Haar measure giving $\mathrm{GL}_2(\mathcal{O}_F)$ measure 1 and equip a maximal torus $T(F)$ in $\mathrm{GL}_2(F)$ with the unique Haar measure giving $T(\mathcal{O}_F)$ measure 1. For any function $f \in C_c^\infty(\mathrm{GL}_2(F))$, denote the orbital integral of f at $\gamma \in T(F)$ by

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) := \int_{T(F) \backslash \mathrm{GL}_2(F)} f(x^{-1}\gamma x) d\bar{x},$$

where $d\bar{x}$ denotes the quotient measure on $T(\mathbb{Q}_p) \backslash \mathrm{G}(E)$ determined by these choices.

In this section we consider the orbital integral $\Phi_{\mathrm{GL}_2(F)}(t, f)$ where $t = \mathrm{diag}(t_1, t_2) \in T_0(\mathcal{O}_F)$. We will evaluate this integral in each of the following cases: (i) f is the unique function with Satake transform given in proposition 2.1; (ii) f is the characteristic function of the Iwahori subgroup \mathcal{I} ; (iii) f is the characteristic function of the maximal compact subgroup K ; (iv) f is the characteristic function of K_+ , where

$$K_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \mid a, d \in 1 + \mathfrak{p}_F \text{ and } b, c \in \mathfrak{p}_F \right\}.$$

To state the result in each case we will need the following definition.

DEFINITION 2.9. For any $\gamma \in \mathrm{GL}_2(F)$, let $\Delta_F(\gamma)$ denote the F -norm of the discriminant of γ ; that is, $\Delta_F(\gamma) = |t_1^{-1}t_2^{-1}(t_1 - t_2)^2|_F^{1/2}$, where t_1 and t_2 are the eigenvalues of γ .

Our main tool in this section is the following lemma.

LEMMA 2.6. *If $t = \mathrm{diag}(t_1, t_2)$ is regular, then*

$$\Phi_{\mathrm{GL}_2(F)}(t, f) = \Delta_F(t)^{-1} \delta_B(t) \int_{\mathrm{U}(F)} \int_K f(k^{-1}tuk) dk du,$$

for any $f \in C_c^\infty(\mathrm{GL}_2(F))$. Here, $\delta_B(t)$ is the modular function $|t_1 t_2^{-1}|_F^{1/2}$ and $\mathrm{U}(F)$ is the group of unipotent upper-triangular matrices in $\mathrm{GL}_2(F)$.

Proof: This elementary lemma follows from the Iwasawa decomposition for $\mathrm{GL}_2(F)$ together with a simple change of variables; since this is a standard result we omit the proof here. \blacksquare

2.2.1. *Case (i): Suppose f is the function with Satake transform given by proposition 2.1*

PROPOSITION 2.2. *Suppose $a_\pi, b_\pi \in \mathbb{N}$, that $a_\pi + b_\pi = g_\pi$ and that $a_\pi \neq b_\pi$. Let s_π be the integer $\frac{r}{g_\pi}$. Let $f_\pi \in \mathcal{H}_F$ be a function whose Satake transform is $\mathcal{S}f_\pi(T_1, T_2) = (p^{r/2})^{g_\pi} (T_1^{s_\pi} + T_2^{s_\pi})^{g_\pi}$. Suppose also that $\gamma_\pi = \mathrm{diag}(\varpi_F^{a_\pi s_\pi}, \varpi_F^{b_\pi s_\pi})$. Then*

$$\Phi_{\mathrm{GL}_2(F)}(\gamma_\pi, f_\pi) = \Delta_F(\gamma_\pi)^{-1} (p^{r/2})^{g_\pi} \binom{g_\pi}{a_\pi},$$

where $\binom{g_\pi}{a_\pi}$ is the binomial coefficient.

Proof: To prove proposition 2.2, we argue as follows. From 2.6 we have

$$\Phi_{\mathrm{GL}_2(F)}(\gamma_\pi, f_\pi) = \Delta_F(\gamma_\pi)^{-1} \delta_B(\gamma_\pi) \int_{\mathrm{U}(F)} \int_K f(k^{-1}\gamma_\pi uk) dk du.$$

Since f_π is the characteristic function of a double- K coset, and since the measure on K has been normalized, the inner integral is just $f_\pi(\gamma_\pi u)$. To

simplify notation slightly, we will write f_k for the coefficient $(f_\pi)_{k, g_\pi - k}$ of $T_1^{ks_\pi} T_2^{(g_\pi - k)s_\pi}$ in $\mathcal{S}f_\pi$ as in definition 2.5. Then

$$\Phi_{\mathrm{GL}_2(F)}(\gamma_\pi, f_\pi) = \Delta_F(\gamma_\pi)^{-1} f_{a_\pi}.$$

To determine f_{a_π} , expand $\mathcal{S}f_\pi(T_1, T_2)$ using the binomial theorem:

$$\begin{aligned} \hat{f}_\pi(T_1, T_2) &= (p^{r/2})^{g_\pi} (T_1^{s_\pi} + T_2^{s_\pi})^{g_\pi} \\ &= (p^{r/2})^{g_\pi} \sum_{k=1}^{g_\pi} \binom{g_\pi}{k} T_1^{ks_\pi} T_2^{(g_\pi - k)s_\pi}. \end{aligned}$$

Thus,

$$f_{a_\pi} = (p^{r/2})^{g_\pi} \binom{g_\pi}{a_\pi}.$$

This proves proposition 2.2. \blacksquare

2.2.2. *Case (ii): Suppose f is the characteristic function of \mathcal{I}*

In this paper, \mathcal{I} always denotes the Iwahori subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \mid a, b, c, d \in \mathcal{O}_F \text{ and } ad - bc \in \mathcal{O}_F^\times \text{ and } c \in \mathfrak{p}_F \right\}.$$

PROPOSITION 2.3. *Let f be the characteristic function of the Iwahori subgroup \mathcal{I} . Suppose that γ is conjugate in $\mathrm{GL}_2(F)$ to a regular element of $T_0(\mathcal{O}_F)$. Then*

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = 2 [K : \mathcal{I}]^{-1} \Delta_F(\gamma)^{-1}.$$

Proof: Since the orbital integral is conjugate-invariant we assume that $\gamma \in T_0(\mathcal{O}_F)$ and then use lemma 2.6 to see that

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} \int_{\mathrm{U}(F)} \int_K f(k^{-1} \gamma u k) dk du.$$

However, f is not K -conjugate-invariant in this case, so we must evaluate the inner integral. Elementary arguments show that

$$\int_K f(k^{-1} \gamma u k) dk = \begin{cases} 0, & \text{if } u \notin \mathrm{U}(\mathcal{O}_F); \\ [K : \mathcal{I}]^{-1}, & \text{if } u \in \mathrm{U}(\mathcal{O}_F^\times); \\ 1, & \text{if } u \in \mathrm{U}(\mathfrak{p}_F). \end{cases}$$

A routine computation now shows that

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} 2[K : \mathcal{I}]^{-1}.$$

This proves proposition 2.3. \blacksquare

2.2.3. *Case (iii): Suppose f is the characteristic function of K*

Recall that

$$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \mid a, b, c, d \in \mathcal{O}_F \text{ and } ad - bc \in \mathcal{O}_F^\times \right\}.$$

PROPOSITION 2.4. *Let f be the characteristic function of K . Suppose that γ is conjugate in $\mathrm{GL}_2(F)$ to a regular element of $T_0(\mathcal{O}_F)$. Then*

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1}.$$

Proof: Use lemma 2.6 again. \blacksquare

2.2.4. *Case (iv): Suppose f is the characteristic function of K_+*

Recall that

$$K_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \mid a, d \in 1 + \mathfrak{p}_F \text{ and } b, c \in \mathfrak{p}_F \right\}.$$

We write $T_0(1 + \mathfrak{p}_F)$ for $T_0(F) \cap K_+$.

PROPOSITION 2.5. *Let f be the characteristic function of K_+ . Suppose that γ is conjugate in $\mathrm{GL}_2(F)$ to a regular element of $T_0(1 + \mathfrak{p}_F)$. Then*

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} [K : K_+]^{-1} (q_F^2 - 1)(q_F - 1).$$

Proof: Without loss of generality, suppose $\gamma \in T_0(1 + \mathfrak{p}_F)$. As above, lemma 2.6 shows that we should consider $\int_K f(k^{-1}\gamma uk) dk$. We see that $f(k^{-1}\gamma uk) \neq 0$ if and only if $f(k^{-1}\gamma uk) = 1$. Since K_+ is normal in K , it follows that $k^{-1}\gamma uk \in K_+$ if and only if $\gamma u \in K_+$. Since $\gamma \in T_0(1 + \mathfrak{p}_F)$ and since $T_0(1 + \mathfrak{p}_F)$ is a subgroup of K_+ , then $\gamma u \in K_+$ if and only if $u \in U(F) \cap K_+ = U(\mathfrak{p}_F)$. Thus,

$$\int_K f(k^{-1}\gamma uk) dk = \begin{cases} 1, & \text{if } u \in U(\mathfrak{p}_F); \\ 0, & \text{otherwise.} \end{cases}$$

Also note that $\delta_B(\gamma) = 1$ in this case. Therefore, by lemma 2.6,

$$\Phi_{\mathrm{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} \int_{\mathrm{U}(\mathfrak{p}_F)} du.$$

Because the measure on $\mathrm{U}(F)$ was chosen so that the measure of $\mathrm{U}(\mathcal{O}_F)$ is 1, the measure of $\mathrm{U}(\mathfrak{p}_F)$ is q_F^{-1} . This proves proposition 2.5. \blacksquare

2.3. Elliptic orbital integrals

DEFINITION 2.10. In this subsection γ denotes a regular elliptic element of $\mathrm{GL}_2(F)$; that is, the centralizer of γ in $\mathrm{GL}_2(F)$ is a maximal torus in $\mathrm{GL}_2(F)$ which is compact modulo the centre of $\mathrm{GL}_2(F)$.

This subsection evaluates the orbital integral $\Phi_{\mathrm{GL}_2(F)}(\gamma, f)$, when γ is regular elliptic, in each of the following cases: (i) f is the characteristic function of \mathcal{I} ; (ii) f is the characteristic function of the Iwahori subgroup K ; (iii) f is the characteristic function of K_+ . To express the results in this section succinctly, we make the following definition.

DEFINITION 2.11. Without loss of generality, for the rest of this section we write

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \alpha \end{pmatrix},$$

and assume that either $\mathrm{ord}_F(\epsilon) = 0$ or $\mathrm{ord}_F(\epsilon) = 1$.

The main tool for this subsection is the affine Bruhat decomposition for $\mathrm{PGL}_2(F)$, given in proposition 2.6 below. Recall first that the affine Weyl group \hat{W} for PGL_2 may be written as a semi-direct product of the Weyl group W for PGL_2 and the free group generated by the Coxeter element in \hat{W} .

DEFINITION 2.12. Let s_1 be a generator for W and let c be the Coxeter element in \hat{W} . In each case below, $\ell(w) = \ell$. (Elements of $\mathrm{PGL}_2(F)$ are denoted with brackets to distinguish them from elements of $\mathrm{GL}_2(F)$.)

- Suppose $w = c^\ell$ with $\ell \geq 0$. Define $\phi_w : \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \rightarrow \mathrm{PGL}_2(F)$ by

$$\phi_w(\xi) := \begin{bmatrix} 1 & 0 \\ \varpi_F^\ell \xi & \varpi_F^\ell \end{bmatrix}.$$

- Suppose $w = c^{\ell+1}s_1$ with $\ell \geq 0$. Define $\phi_w : \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \rightarrow \mathrm{PGL}_2(F)$ by

$$\phi_w(\xi) := \begin{bmatrix} 0 & 1 \\ \varpi_F^{\ell+1} & \varpi_F \xi \end{bmatrix}.$$

- Suppose $w = c^{-\ell}$ with $\ell \geq 1$. Define $\phi_w : \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \rightarrow \mathrm{PGL}_2(F)$ by

$$\phi_w(\xi) := \begin{bmatrix} \varpi_F^\ell & \xi \\ 0 & 1 \end{bmatrix}.$$

- Suppose $w = c^{1-\ell}s_1$ with $\ell \geq 1$. Define $\phi_w : \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \rightarrow \mathrm{PGL}_2(F)$ by

$$\phi_w(\xi) := \begin{bmatrix} \xi & \varpi_F^{\ell-1} \\ 1 & 0 \end{bmatrix}.$$

PROPOSITION 2.6. *Let \mathcal{I}' be the image of \mathcal{I} in $\mathrm{PGL}_2(F)$, and let $\mathcal{B}(w)$ be the image of $\mathcal{O}_F/\mathfrak{p}_F^{\ell(w)}$ under $\mathcal{O}_F/\mathfrak{p}_F \xrightarrow{\phi_w} \mathrm{PGL}_2(F) \rightarrow \mathrm{PGL}_2(F)/\mathcal{I}'$, with $\ell(w)$ given in definition 2.12. Then $\mathrm{PGL}_2(F)/\mathcal{I}'$ is the disjoint union of Bruhat cells $\mathcal{B}(w)$ as w runs through the affine Weyl group \hat{W} for PGL_2 .*

For later reference, the following list describes $\mathrm{Inn} \phi_w(\xi)^{-1}\gamma \in \mathrm{GL}_2(F)$, where γ is given in definition 2.11. Here, $\mathrm{Inn} : \mathrm{PGL}_2(F) \rightarrow \mathrm{AutGL}_2(F)$ is defined by $\mathrm{Inn}(g')\gamma = g'^{-1}\gamma g$, where g' denotes the image of $g \in \mathrm{GL}_2(F)$ under $\mathrm{GL}_2(F) \rightarrow \mathrm{PGL}_2(F)$. Because this list will be invoked in the proof of each proposition in this subsection, we artificially elevate the list to the status of a lemma.

LEMMA 2.7. *Recall the notation from proposition 2.6.*

- Suppose $w = c^\ell$ with $\ell \geq 0$. Then

$$\mathrm{Inn} \phi_w(\xi)^{-1}\gamma = \begin{pmatrix} \alpha + \beta\xi\varpi_F & \beta\varpi_F^\ell \\ \beta(\epsilon - \varpi_F^2\xi^2)\varpi_F^{-\ell} & \alpha - \beta\xi\varpi \end{pmatrix}.$$

- Suppose $w = c^{\ell+1}s_1$ with $\ell \geq 0$. Then

$$\mathrm{Inn} \phi_w(\xi)^{-1}\gamma = \begin{pmatrix} \alpha - \beta\xi\varpi_F & \beta(\epsilon - \varpi_F^2\xi^2)\varpi_F^{-\ell-1} \\ \beta\varpi_F^{\ell+1} & \alpha + \beta\xi\varpi \end{pmatrix}.$$

• Suppose $w = c^{\ell+1}s_1$ with $\ell \geq 0$. Then

$$\text{Inn } \phi_w(\xi)^{-1}\gamma = \begin{pmatrix} \alpha - \epsilon\beta\xi & \beta(1 - \epsilon\xi^2)\varpi_F^{-\ell} \\ \epsilon\beta\varpi_F^\ell & \alpha + \epsilon\beta\xi \end{pmatrix}.$$

• Suppose $w = c^{1-\ell}s_1$ with $\ell \geq 1$. Then

$$\text{Inn } \phi_w(\xi)^{-1}\gamma = \begin{pmatrix} \alpha + \epsilon\beta\xi & \epsilon\beta\varpi_F^{\ell-1} \\ \beta(1 - \epsilon\xi^2)\varpi_F^{1-\ell} & \alpha - \epsilon\beta\xi \end{pmatrix}.$$

Finally we are ready to evaluate the orbital integral $\Phi_{\text{GL}_2(F)}(\gamma, f)$, by considering the three cases described in the beginning of this subsection.

2.3.1. *Case (i): Suppose f is the characteristic function of \mathcal{I}*

PROPOSITION 2.7. *Let f be the characteristic function of the Iwahori subgroup \mathcal{I} and suppose that γ is a regular elliptic element in \mathcal{I} . Then*

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} [K : \mathcal{I}]^{-1} c_F(\gamma),$$

where

$$c_F(\gamma) = \begin{cases} 2(q_F - 1)^{-1} [2|_F(1 + q_F) - (1 + q_F)\Delta_F(\gamma)], & \text{if } \text{ord}_F(\epsilon) = 0; \\ 2(q_F - 1)^{-1} [2|_F q_F^{1/2} - (1 + q_F)\Delta_F(\gamma)], & \text{if } \text{ord}_F(\epsilon) = 1. \end{cases}$$

Proof: To prove proposition 2.7 we return to the definition of the orbital integral. Let Z denote the center of the group GL_2 , and let $T' := T/Z$. Since $Z(F)$ is a normal subgroup of $T(F)$, the function $g \mapsto f(g^{-1}\gamma g)$ on $T(F)\backslash\text{GL}_2(F)$ factors through the natural projection to give a function $g \mapsto f(\text{Inn}(g)^{-1}\gamma)$ on $T'(F)\backslash\text{PGL}_2(F)$. Equip $Z(F)$ with the unique Haar measure such that the measure of $Z(\mathcal{O}_F)$ is 1 and indicate this measure by dz . Then

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \int_{T'(F)\backslash\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'\bar{g},$$

where $d'\bar{g}$ is the quotient measure. Since $T(F)$ is compact modulo $Z(F)$, the measure of $T'(F)$ is finite. Thus,

$$\int_{T'(F)\backslash\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'\bar{g} = \text{meas}(T'(F))^{-1} \int_{\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'g.$$

In fact, $\text{meas}(T'(F)) = 1$, since $T'(F) = T(\mathcal{O}_F)/Z(\mathcal{O}_F)$. Therefore,

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \int_{\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'g.$$

Recall that \mathcal{I}' is the image of \mathcal{I} under the quotient map $\text{GL}_2(F) \rightarrow \text{PGL}_2(F)$. (Note that $\mathcal{I}' = \mathcal{I}/Z(F) = \mathcal{I}/Z(\mathcal{O}_F)$.) Since \mathcal{I}' is compact, standard tricks with quotient measures give

$$\int_{\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'g = \int_{\text{PGL}_2(F)/\mathcal{I}'} \int_{\mathcal{I}'} f(\text{Inn}(yh)^{-1}\gamma) d'h d'\bar{y}.$$

Since f is bi- \mathcal{I}' -invariant, it follows that

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \text{meas}(\mathcal{I}') \int_{\mathcal{B}} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y},$$

where $\mathcal{B} := \text{PGL}_2(F)/\mathcal{I}'$. By our choice of measures, $\text{meas}(\mathcal{I}') = [K : \mathcal{I}]^{-1}$.

With functions $\phi_w : \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \rightarrow \text{PGL}_2(F)$ defined at the beginning of this section, define

$$\mathcal{B}(w) = \left\{ \phi_w(\xi)\mathcal{I}' \in \mathcal{B} \mid \xi \in \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \right\}.$$

Now, let \mathcal{B}_γ be the set of all Iwahori subgroups containing $\gamma \in \text{PGL}_2(F)$. For each $w \in \hat{W}$, define $\mathcal{B}(w)_\gamma = \mathcal{B}(w) \cap \mathcal{B}_\gamma$. Thus,

$$|\mathcal{B}(w)_\gamma| = \left| \left\{ \xi \in \mathcal{O}_F/\mathfrak{p}_F^{\ell(w)} \mid \text{Inn} \phi_w(\xi)^{-1}\gamma \in \mathcal{I} \right\} \right|.$$

We now use lemma 2.7 to calculate $|\mathcal{B}(w)_\gamma|$.

- Suppose $w = c^\ell$, where $\ell \geq 0$; then $\text{Inn} \phi_w(\xi)^{-1}\gamma \in \mathcal{I}'$ if and only if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1$. Thus, in this case, $|\mathcal{B}(w)_\gamma| = q_F^\ell$ if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1$ and $|\mathcal{B}(w)_\gamma| = 0$ otherwise.

- Suppose $w = c^{\ell+1}s_1$, where $\ell \geq 0$; then $\text{Inn} \phi_w(\xi)^{-1}\gamma \in \mathcal{I}'$ if and only if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1$. Thus, in this case $|\mathcal{B}(w)_\gamma| = q_F^\ell$ if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1$ and $|\mathcal{B}(w)_\gamma| = 0$ otherwise.

- Suppose $w = c^{-\ell}$, where $\ell \geq 1$; then $\text{Inn} \phi_w(\xi)^{-1}\gamma \in \mathcal{I}'$ if and only if $1 \leq \ell \leq \text{ord}_F(\beta)$. Thus, in this case, $|\mathcal{B}(w)_\gamma| = q_F^\ell$ if $1 \leq \ell \leq \text{ord}_F(\beta)$ and $|\mathcal{B}(w)_\gamma| = 0$ otherwise.

• Suppose $w = c^{1-\ell}s_1$, where $\ell \geq 1$; then $\text{Inn } \phi_w(\xi)^{-1}\gamma \in \mathcal{I}'$ if and only if $1 \leq \ell \leq \text{ord}_F(\beta)$, so $|\mathcal{B}(w)_\gamma| = q_F^\ell$ if $1 \leq \ell \leq \text{ord}_F(\beta)$ and $|\mathcal{B}(w)_\gamma| = 0$ otherwise.

Gathering these four cases together we find:

$$\begin{aligned}
 \sum_{w \in \tilde{W}} |\mathcal{B}(w)_\gamma| &= \sum_{0 \leq \ell \in \mathbb{Z}} |\mathcal{B}(c^\ell)_\gamma| + \sum_{0 \leq \ell \in \mathbb{Z}} |\mathcal{B}(c^{1+\ell}s_1)_\gamma| \\
 &\quad + \sum_{1 \leq \ell \in \mathbb{Z}} |\mathcal{B}(c^{-\ell})_\gamma| + \sum_{1 \leq \ell \in \mathbb{Z}} |\mathcal{B}(c^{1-\ell}s_1)_\gamma| \\
 &= \sum_{\ell=0}^{\text{ord}_F(\epsilon\beta)-1} q_F^\ell + \sum_{\ell=0}^{\text{ord}_F(\epsilon\beta)-1} q_F^\ell \\
 &\quad + \sum_{\ell=1}^{\text{ord}_F(\beta)} q_F^\ell + \sum_{\ell=1}^{\text{ord}_F(\beta)} q_F^\ell \\
 &= 2(q_F - 1)^{-1} [(q_F^{\text{ord}_F(\epsilon\beta)} - 1) + q_F(q_F^{\text{ord}_F(\beta)} - 1)] \\
 &= 2(q_F - 1)^{-1} [q_F^{\text{ord}_F(\beta)+1} + q_F^{\text{ord}_F(\epsilon\beta)} - q_F - 1].
 \end{aligned}$$

Now, return to the integral

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = [K : \mathcal{I}]^{-1} \int_{\mathcal{B}} f(\text{Inn}(y)^{-1}\gamma) d'y,$$

and note that the integral above is precisely the (counting) measure of the affine Springer fibre above γ ; that is,

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = [K : \mathcal{I}]^{-1} |\mathcal{B}_\gamma|.$$

It only remains to phrase the result as in the statement of the proposition.

Recall that $\text{ord}_F(\epsilon)$ is either 0 or 1. If $\text{ord}_F(\epsilon) = 0$ then

$$\Delta_F(\gamma) = |2|_F q_F^{-\text{ord}_F(\beta)}$$

and

$$|\mathcal{B}_\gamma| = 2(q_F - 1)^{-1} [(1 + q_F)(q_F^{\text{ord}_F(\beta)} - 1)].$$

Thus,

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} 2[K : \mathcal{I}]^{-1} (q_F - 1)^{-1} [(1 + q_F)|2|_F - \Delta_F(\gamma)].$$

On the other hand, if $\text{ord}_F(\epsilon) = 1$ then

$$\Delta_F(\gamma) = |2|_F q_F^{-\text{ord}_F(\beta)} q_F^{-1/2}$$

and

$$|\mathcal{B}_\gamma| = 2(q_F - 1)^{-1} [2q_F^{\text{ord}_F(\beta)+1} - q_F - 1].$$

Thus,

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} 2[K : \mathcal{I}]^{-1} (q_F - 1)^{-1} [2|2|_F q_F^{1/2} - (1 + q_F)\Delta_F(\gamma)].$$

This completes the proof of proposition 2.7. \blacksquare

2.3.2. Case (ii): Suppose f is the characteristic function of K

PROPOSITION 2.8. *Let f be the characteristic function of K and suppose that γ is a regular elliptic element in K . Then*

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} c_F(\gamma),$$

where

$$c_F(\gamma) = \begin{cases} (q_F - 1)^{-1} [(1 + q_F)|2|_F - 2\Delta_F(\gamma)], & \text{if } \text{ord}_F(\epsilon) = 0; \\ (q_F - 1)^{-1} [2q_F^{1/2}|2|_F - 2\Delta_F(\gamma)], & \text{if } \text{ord}_F(\epsilon) = 1. \end{cases}$$

Proof: To prove proposition 2.8, use lemma 2.7 as in the proof of proposition 2.7 to show that

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = (q_F - 1)^{-1} [q_F^{\text{ord}_F(\beta)+1} + q_F^{\text{ord}_F(\epsilon\beta)} - 2].$$

Then, re-write the result according to the form of proposition 2.8. Since this result can also be found in [L], where it is proved by a calculation in the affine Bruhat-Tits building, we omit the details here. This result can also be proved using the affine Grassmannian for PGL_2 . \blacksquare

2.3.3. Case (iii): Suppose f is the characteristic function of K_+

PROPOSITION 2.9. *Let f be the characteristic function of K_+ and suppose that γ is a regular elliptic element of K_+ . Then*

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} [K : K_+]^{-1} (q_F^2 - 1)(q_F - 1) c_F(\gamma),$$

where

$$c_F(\gamma) = \begin{cases} (q_F - 1)^{-1} [2|_F(1 + q_F) - 2q_F \Delta_F(\gamma)], & \text{if } \text{ord}_F(\epsilon) = 0; \\ (q_F - 1)^{-1} [2|_F(q_F^{1/2} + q_F^{3/2}) - 2q_F \Delta_F(\gamma)] & \text{if } \text{ord}_F(\epsilon) = 1. \end{cases}$$

Proof: As in the proof of proposition 2.7, we re-write the orbital integral as follows:

$$\begin{aligned} \Phi_{\text{GL}_2(F)}(\gamma, f) &= \int_{T'(F) \backslash \text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'\bar{g} \\ &= \int_{\text{PGL}_2(F)} f(\text{Inn}(g)^{-1}\gamma) d'g \\ &= \int_{\text{PGL}_2(F)/\mathcal{I}'} \int_{\mathcal{I}'} f(\text{Inn}(yh)^{-1}\gamma) d'h d'\bar{y}. \end{aligned}$$

Although f is not bi- \mathcal{I} -invariant, it is \mathcal{I} -conjugate invariant since K_+ is a normal subgroup of K and $K_+ \subset \mathcal{I} \subset K$. Thus

$$\begin{aligned} \Phi_{\text{GL}_2(F)}(\gamma, f) &= \text{meas}(\mathcal{I}') \int_{\text{PGL}_2(F)/\mathcal{I}'} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y} \\ &= (q_F + 1)^{-1} \int_{\mathcal{B}} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y} \\ &= (q_F + 1)^{-1} \sum_{w \in \tilde{W}} \int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y}, \end{aligned}$$

where we have use the notation of the proof of proposition 2.7. To calculate $\int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y}$ we use lemma 2.7 again.

• Suppose $w = c^\ell$, where $\ell \geq 0$; then $\text{Inn } \phi_w(\xi)^{-1}\gamma \in K_+$ if and only if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1$. Thus, in this case,

$$\int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y} = \begin{cases} q_F^\ell, & \text{if } 0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 1; \\ 0, & \text{otherwise.} \end{cases}$$

• Suppose $w = c^{\ell+1}s_1$, where $\ell \geq 0$; then $\text{Inn } \phi_w(\xi)^{-1}\gamma \in K_+$ if and only if $0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 2$. Thus, in this case

$$\int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'\bar{y} = \begin{cases} q_F^\ell, & \text{if } 0 \leq \ell \leq \text{ord}_F(\epsilon\beta) - 2; \\ 0, & \text{otherwise.} \end{cases}$$

• Suppose $w = c^{-\ell}$, where $\ell \geq 1$; then $\text{Inn } \phi_w(\xi)^{-1}\gamma \in K_+$ if and only if $1 \leq \ell \leq \text{ord}_F(\beta) - 1$. Thus, in this case,

$$\int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'y = \begin{cases} q_F^\ell, & \text{if } 0 \leq \ell \leq \text{ord}_F(\beta) - 1; \\ 0, & \text{otherwise.} \end{cases}$$

• Finally, suppose $w = c^{1-\ell}s_1$, where $\ell \geq 1$; then $\text{Inn } \phi_w(\xi)^{-1}\gamma \in K_+$ if and only if $1 \leq \ell \leq \text{ord}_F(\beta)$. Thus, in this case,

$$\int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'y = \begin{cases} q_F^\ell, & \text{if } 0 \leq \ell \leq \text{ord}_F(\beta); \\ 0, & \text{otherwise.} \end{cases}$$

Gathering these four cases together we find:

$$\begin{aligned} (q_F + 1)\Phi_{\text{GL}_2(F)}(\gamma, f) &= \sum_{w \in \hat{W}} \int_{\mathcal{B}(w)} f(\text{Inn}(y)^{-1}\gamma) d'y \\ &= \sum_{\ell=0}^{\text{ord}_F(\epsilon\beta)-1} q_F^\ell + \sum_{\ell=0}^{\text{ord}_F(\epsilon\beta)-2} q_F^\ell \\ &\quad + \sum_{\ell=1}^{\text{ord}_F(\beta)-1} q_F^\ell + \sum_{\ell=1}^{\text{ord}_F(\beta)} q_F^\ell \\ &= (q_F - 1)^{-1}(q_F + 1)[q_F^{\text{ord}_F(\epsilon\beta)} + q_F^{\text{ord}_F(\epsilon\beta)-1} - 2]. \end{aligned}$$

It only remains to phrase the result as in the statement of the proposition. Observe that $[K : K_+] = q_F(q_F + 1)(q_F - 1)^2 = q_F(q_F^2 - 1)(q_F - 1)$.

Recall that $\text{ord}_F(\epsilon)$ is either 0 or 1. If $\text{ord}_F(\epsilon) = 0$ then

$$\Delta_F(\gamma) = |2|_F q_F^{-\text{ord}_F(\beta)}$$

and

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = (q_F - 1)^{-1}[q_F^{\text{ord}_F(\beta)} + q_F^{\text{ord}_F(\beta)-1} - 2].$$

It follows that

$$\Phi_{\text{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} [K : K_+]^{-1}(q_F^2 - 1)[|2|_F(1 + q_F) - 2q_F\Delta_F(\gamma)].$$

On the other hand, if $\text{ord}_F(\epsilon) = 1$ then

$$\Delta_F(\gamma) = |2|_F q_F^{-\text{ord}_F(\beta)} q_F^{-1/2}$$

and

$$\Phi_{\mathbf{GL}_2(F)}(\gamma, f) = (q_F - 1)^{-1} [q_F^{\text{ord}_F(\beta)+1} + q_F^{\text{ord}_F(\beta)} - 2].$$

It follows that

$$\Phi_{\mathbf{GL}_2(F)}(\gamma, f) = \Delta_F(\gamma)^{-1} [K : K_+]^{-1} (q_F^2 - 1) [2|_F(q_F^{1/2} + q_F^{3/2}) - 2q_F \Delta_F(\gamma)].$$

This completes the proof of proposition 2.9. \blacksquare

3. EVALUATION OF A GLOBAL ORBITAL INTEGRAL

We now apply the techniques of sections 2.1 and 2 to the objects of section 1. A little more notation is necessary before we can state precisely our main result. Throughout this discussion, λ will range over primes of L lying over the rational prime l ; π and p will enjoy the same relation. If v is any place of \mathbb{Q} – be it archimedean or non-archimedean – there is a canonical isomorphism $\mathbb{Q}_v \otimes_{\mathbb{Z}} \mathcal{O}_L \cong \bigoplus_{w|v} L_w$, where the subscripts v and w denote completions. This isomorphism preserves integrality: if v is a finite unramified place then $\mathbb{Z}_v \otimes_{\mathbb{Z}} \mathcal{O}_L \cong \bigoplus_{w|v} \mathcal{O}_{L,w}$. The decomposition induces a canonical decomposition on $\mathbb{Z}_v \otimes_{\mathbb{Z}} \mathcal{O}_L$ -objects. We will often indicate this decomposition through judicious use of subscripts. For instance, the isomorphisms above induce an isomorphism between $\mathbf{G}(\mathbb{Q}_l)$, itself isomorphic to $\mathbf{GL}_2(\mathbb{Q}_l \otimes \mathcal{O}_L)$, and $\prod_{\lambda|l} \mathbf{GL}_2(L_\lambda)$. If $x \in \mathbf{G}(\mathbb{Q}_l)$, we write x_λ for the image of x under the projection to $\mathbf{GL}_2(L_\lambda)$.

As in the first section, let g_π denote the residue degree of L at π . In the proof of the main theorem we will carefully describe the p -divisible group $X[p^\infty]$ of an \mathcal{O}_L -abelian variety X . For now, we content ourselves with the following declarations.

DEFINITION 3.1. We will say that X has no supersingular part if $X[p^\infty]$ has no factors isogenous to the p -divisible group of a supersingular elliptic curve.

In 3.2 we will see that the formal isogeny type of $X[p^\infty]$ is determined by data $\{a_\pi\}_{\pi|p}$, where each a_π is between zero and $\frac{g_\pi}{2}$. Moreover, our assumption that X has no supersingular part implies that a_π is, in fact, an integer. The slopes of the Newton polygon of X are then $\{\frac{a_\pi}{g_\pi}, 1 - \frac{a_\pi}{g_\pi}\}_{\pi|p}$. The abelian variety is ordinary if all a_π are zero.

THEOREM 3.1. *Suppose $(X, \iota, \alpha) \in \mathcal{M}(k)$ has no supersingular part. Thus, X is a g -dimensional abelian variety over a field with p^r elements,*

equipped with an action ι by the integers of a totally real field L and a $\Gamma_0(N_0, N)$ -level structure α , where p , N_0 , N and $\text{disc}(L, \mathbb{Q})$ are pairwise relatively prime and N_0N is square-free. Then the cardinality $|\mathcal{Y}_{(X, \iota, \alpha)}(k)|$ of the isogeny class of (X, ι, α) is

$$p^{\frac{r}{2}g} h(M) \frac{[\mathbf{G}(\widehat{\mathbb{Z}}^p) : \Gamma(N)]}{\text{Norm}(N\mathcal{O}_L)} 2^{e(L, N_0)} C(\gamma, N_0, N) \prod_{w|\infty} \Delta_w(\gamma) \prod_{\pi|p} \binom{g_\pi}{a_\pi}$$

where:

- $\mathbf{G} \cong \text{res}_{\mathcal{O}_L, \mathbb{Z}} \mathbf{GL}_2$, and $\gamma \in \mathbf{G}(\widehat{\mathbb{Z}}_{\text{fin}}^p)$ is associated to X as in section 1;
- $h(M)$ is the class number of the ring $M = \text{End}_{\mathcal{O}_L}(X) \otimes \mathbb{Q}$;
- $e(L, N_0)$ is the number of primes λ dividing N_0 ;
- $\Delta_w(\gamma)$ is the norm of the discriminant of γ_w as in 2.9;
- $\{a_\pi\}_{\pi|p}$ determine the isogeny class of $X[p^\infty]$ as in 3.2, and $\binom{g_\pi}{a_\pi}$ is the binomial coefficient;
- $C(\gamma, N_0, N) = \prod_{l \neq p} \prod_{\lambda|l} C_\lambda(\gamma, N_0, N)$, where $\|\lambda\|$ denotes the cardinality of the residue field at λ , S is the set of all finite places λ of L such that $\text{ord}_\lambda(\text{disc}(M, L)) \equiv 1 \pmod{2}$, and

$$C_\lambda(\gamma, N_0, N) = \begin{cases} 1 & \gamma_\lambda \text{ split in } \mathbf{GL}_2(L_\lambda) \\ \frac{(1+\|\lambda\|)|2|_\lambda - 2\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N_0N \in \mathcal{O}_{L_\lambda}^\times \\ \frac{2|2|_\lambda \|\lambda\|^{1/2} - \Delta_\lambda(\gamma)(1+\|\lambda\|)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N_0 \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{(1+\|\lambda\|)(|2|_\lambda - \Delta_\lambda(\gamma))}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N_0 \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda \|\lambda\|^{1/2} - 2\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N_0N \in \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda(1+\|\lambda\|) - 2\|\lambda\|\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \notin S, \text{ and } N \notin \mathcal{O}_{L_\lambda}^\times \\ \frac{|2|_\lambda(\|\lambda\|^{1/2} + \|\lambda\|^{3/2}) - 2\|\lambda\|\Delta_\lambda(\gamma)}{\|\lambda\| - 1} & \gamma_\lambda \text{ elliptic, } \lambda \in S, \text{ and } N \notin \mathcal{O}_{L_\lambda}^\times \end{cases}$$

Proof: Proposition 1.1 expresses the cardinality of the isogeny class of (X, ι, α) as a product of a global volume and local integrals on \mathbf{G} . The initial term is readily identified as $\text{vol}(T(\mathbb{Q}) \backslash T(\mathbb{A}_{\text{fin}})) = h(M)$. We will rewrite the integrals on \mathbf{G} as local integrals on \mathbf{GL}_2 , and then invoke the results of section 2. It is convenient to discuss the l -adic and p -adic integrals separately.

We first consider the prime-to- p integrals. Suppose that l is a rational prime other than p . By the decomposition discussed above,

$$\Phi_{\mathbf{G}(\mathbb{Q}_l)}(\gamma, fi) = \prod_{\lambda|l} \Phi_{\mathbf{GL}_2(L_\lambda)}(\gamma, f_\lambda).$$

Here, f_λ is the characteristic function of $K(N_0, N)_\lambda$ divided by the index $[K_\lambda : K(N_0, N)_\lambda]$. As remarked in section one, $K(N_0, N)_\lambda$ is either the standard Iwahori subgroup of $\mathbf{GL}_2(\mathcal{O}_\lambda)$, the kernel of $\mathbf{GL}_2(\mathcal{O}_\lambda) \rightarrow \mathbf{GL}_2(\mathcal{O}_\lambda/\lambda\mathcal{O}_\lambda)$, or $\mathbf{GL}_2(\mathcal{O}_\lambda)$, depending on whether λ divides N_0 , divides N , or is relatively prime to both. (The reader should not be alarmed by the weighted characteristic functions; they are chosen to compensate for the choice of measure in section 2.) Define $C_\lambda(\gamma, N_0, N)$ as above; let $e(\lambda, N) = 1$ if $\lambda|N$, and 0 otherwise; and let $e(l, N) = \sum_{\lambda|l} e(\lambda, N)$.

We now invoke the appropriate proposition of section 2 for each $F = L_\lambda$, depending on the interaction between λ , N and γ . Specifically, when γ is split in $\mathbf{GL}_2(L_\lambda)$, apply proposition 2.3, 2.5 or 2.4 (as λ divides N_0 , N or neither) to see that

$$\Phi_{\mathbf{GL}_2(L_\lambda)}(\gamma, f_\lambda) = \frac{[\mathbf{G}(\widehat{\mathbb{Z}}^p)_\lambda : \Gamma(N)]}{\|\lambda\|^{e(\lambda, N)}} 2^{e(\lambda, N_0)} \Delta_\lambda(\gamma)^{-1} C_\lambda(\gamma, N_0, N).$$

Similarly, when γ is elliptic in $\mathbf{GL}_2(L_\lambda)$, apply proposition 2.7, 2.9 or 2.8 (as λ divides N_0 , N or neither) to produce the same result. Then

$$\Phi_{\mathbf{G}(\mathbb{Q}_l)}(\gamma, fl) = \prod_{\lambda|l} \Phi_{\mathbf{GL}_2(L_\lambda)}(\gamma, f_\lambda) = \prod_{\lambda|l} 2^{e(\lambda, N)} \Delta_\lambda(\gamma)^{-1} C_\lambda(\gamma, N_0, N).$$

Performing this calculation at every $l \neq p$ while recalling that $p \nmid N_0 N$, we see that

$$\Phi_{\mathbf{G}(\mathbb{A}_{\text{fin}}^p)}(\gamma, f) = \frac{[\mathbf{G}(\widehat{\mathbb{Z}}^p) : \Gamma(N)]}{\text{Norm}(N\mathcal{O}_L)} 2^{e(L, N)} \prod_{\lambda \nmid p} \Delta_\lambda(\gamma)^{-1} C_\lambda(\gamma, N_0, N).$$

Now we consider the integral at p . The σ -conjugacy class of δ exactly determines the (formal) isogeny type of $X[p^\infty]$ as an \mathcal{O}_L -module. In fact, $H := H_1^{\text{cris}}(X, E)$ may be regarded as the Dieudonné module of $X[p^\infty]$. The isogeny type of $X[p^\infty]$ is given by its Newton polygon, a combinatorial device which encodes the p -adic eigenvalues of Φ acting on H . Since X admits a polarization, $X[p^\infty]$ admits a quasipolarization, and the F-isocrystal H is self-dual. (Unless otherwise specified, all F-isocrystals are σ -F-isocrystals.)

Both \mathcal{O}_L and \mathbb{Z}_p act on $X[p^\infty]$; thus, so does $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \oplus_{\pi|p} \mathcal{O}_{L, \pi}$. There is in fact an \mathcal{O}_L -linear quasipolarization of H , and we get a decomposition $(H, \Phi) \cong \oplus_{\pi|p} (H_\pi, \Phi_\pi)$ of H as a sum of self-dual F-isocrystals. Moreover, each H_π is a free, rank-two module over $L_\pi \otimes E \cong \oplus_{\iota \in \text{Hom}(L_\pi, E)} L_\pi \otimes_\iota E$. Recall from section 2.1 that $\mathbf{G}_\pi = \text{res}_{\mathcal{O}_{L_\pi, \mathbb{Z}_p}} \mathbf{GL}_2$. As at the beginning of

the present section, there is a canonical isomorphism $G(E) \cong \prod_{\pi|p} G_\pi(E)$. We denote the π -adic component of δ by δ_π and analyze the σ -conjugacy class of δ_π in G_π .

Now, (H_π, Φ_π) itself has a slope decomposition, $H_\pi = \bigoplus_{i=1}^t H_\pi(\lambda_\pi(i))$, where the $\lambda_\pi(i)$ are distinct rational numbers between zero and one.

LEMMA 3.1. *The F -isocrystal (H_π, Φ_π) has exactly two distinct slopes.*

Proof: This is standard. We avail ourselves of common facts about Dieudonné modules; see [M] or [D]. Let $D_{\lambda_\pi(i)}$ be the central simple \mathbb{Q}_p -algebra of invariant $\lambda_\pi(i)$. Let $d_\pi(i)$ be the smallest natural number so that $d_\pi(i) \cdot \lambda_\pi(i) \in \mathbb{Z}$. Then $\mu(\lambda_\pi(i)) := (\dim_E H_\pi(\lambda_\pi(i)))/d_\pi(i)$ is an integer; by the autoduality of H_π , $\mu(\lambda_\pi(i)) = \mu(1 - \lambda_\pi(i))$ and $\text{End}(H_\pi, \Phi_\pi) \cong \bigoplus_{i=1}^t \text{Mat}_{\mu(\lambda_\pi(i))}(D_{\lambda_\pi(i)})$. The field L_π is a commutative \mathbb{Q}_p -algebra of rank $\frac{1}{2} \dim_E H_\pi$, and 1_{L_π} must act as the identity on H_π . We deduce that either there are exactly two distinct slopes, $\lambda_\pi := \frac{a_\pi}{g_\pi}$ and $1 - \lambda_\pi := \frac{b_\pi}{g_\pi}$; or there is exactly one slope, $\frac{1}{2}$. The latter case having been excluded – remember, X has no supersingular part – the former case must hold. ■

DEFINITION 3.2. By way of normalization, we henceforth assume that $a_\pi < b_\pi$. Thus, $(H, \Phi) \cong \bigoplus_{\pi|p} (H_\pi, \Phi_\pi)$, and the slopes of the Newton polygon of (H_π, Φ_π) are $\frac{a_\pi}{g_\pi}$ and $\frac{b_\pi}{g_\pi}$.

Given this, we can be fairly explicit about δ and its norms. Let $s_\pi = \frac{r}{g_\pi}$; by hypothesis, this is an integer.

LEMMA 3.2. *Recall that $\xi_\pi = N_\pi(\delta_\pi)$ and $\gamma_\pi = N_{E/L_\pi}(\xi_\pi)$. Then γ_π is $\mathbf{GL}_2(L_\pi)$ -conjugate to $\text{diag}(p^{a_\pi s_\pi}, p^{b_\pi s_\pi})$.*

Proof: While it is possible to proceed with an analysis of σ -conjugacy along the lines of section 4 of [L], we content ourselves with the following basic computations. In the notation of section 2.1, let $H_\pi^{(1)}$ be the ρ_1 -eigenspace $H_\pi \otimes_{L_\pi, \rho_1} E$. Then $\xi \circ \tau$ corresponds to the Frobenius of the rank two τ - F -isocrystal $(H_\pi^{(1)}, \Phi_\pi^{2g_\pi})$. The proof of 1.4.1 of [Ka] shows that ξ_π is τ -conjugate to

$$\begin{pmatrix} p^{a_\pi} & * \\ 0 & p^{b_\pi} \end{pmatrix},$$

and γ_π is conjugate to $\text{diag}(p^{a_\pi s_\pi}, p^{b_\pi s_\pi})$. ■

Armed with these observations we invoke proposition 2.2 and easily compute the relevant twisted orbital integral.

$$\begin{aligned}\Phi_{G(E)}^\sigma(\delta, \varphi) &= \prod_{\pi|p} \Phi_{G_\pi(E)}^\sigma(\delta_\pi, \varphi_\pi) \\ &= \prod_{\pi|p} \binom{g_\pi}{a_\pi} p^{\frac{g}{2}g_\pi} \Delta_\pi(\gamma)^{-1}.\end{aligned}$$

We now combine the l -adic and p -adic integrals to produce the global result. Starting with proposition 1.1, we see that

$$\begin{aligned}|\mathcal{Y}_{(X,\iota,\alpha)}(k)| &= \text{vol}(T(\mathbb{Q}) \backslash T(\mathbb{A}_{\text{fin}})) \Phi_{G(\mathbb{A}_{\text{fin}}^p)}(\gamma, f_{\text{fin}}^p) \Phi_{G(E)}^\sigma(\delta, \phi) \\ &= h(M)v(N)C(\gamma, N) \left(\prod_{\pi|p} \binom{g_\pi}{a_\pi} \right) p^{\frac{g}{2}g} \prod_{\lambda \text{ finite}} \Delta_\lambda(\gamma)^{-1}.\end{aligned}$$

By the product formula, this last term is equal to $\prod_{w|\infty} \Delta_w(\gamma)$. This concludes the proof of theorem 3.1. \blacksquare

4. ASYMPTOTICS

We would like to briefly mention the asymptotics of the following situation. Fix a k -point $(X/k, \iota, \alpha)$ as above. Denote by k_t an extension of k of degree t . We will study the dependence of $|\mathcal{Y}_{(X,\iota,\alpha)}(k_t)|$, the cardinality of the k_t -isogeny class of X , on t . Let γ_t be the conjugacy class in $G(\mathbb{A}_{\text{fin}}^p)$ associated to the Frobenius of $X \times \text{Spec } k_t$.

PROPOSITION 4.1. *Suppose $(X, \iota, \alpha) \in \mathcal{M}(k)$ has no supersingular part. Then there are constants c and c' , depending only on g , N_0 and N , so that $c' < C(\gamma_t, N_0, N) < c(\log \log q^t \log \log \log q^t)^2$. Let $\chi(t) = 2^{-g} \prod_{w|\infty} \Delta_w(\gamma_t)$. Then $0 < \chi(t) < 1$, and $\chi(t)$ has a positive expected value depending only on X .*

Proof: The existence of the desired c' is simple to deduce from the definition of the local factors $C(\gamma_t, N_0, N)$. Observe that only those λ which are in S or which lie over 2 have $C_\lambda(\gamma_t, N_0, N) < 1$ and, using the fact that

$0 < \Delta_\lambda(\gamma_t) \leq 1$, bound each one separately. Finding an upper bound for $C(\gamma, N_0, N)$ is only slightly more involved. As an initial simplification, let

$$D(\gamma, N_0, N) = \prod_{\gamma_\lambda \text{ elliptic}, N \notin \mathcal{O}_{L_\lambda}^\times} \|\lambda\|^{-1}.$$

Since there are bounds $[G(\widehat{\mathbb{Z}}^p) : \Gamma(N)]^{-1} \leq D(\gamma, N_0, N) \leq 1$ independent of γ , we may (and do) replace $C(\gamma, N_0, N)$ by $D(\gamma, N_0, N)C(\gamma, N_0, N)$. Considering the various possibilities for $C_\lambda(\gamma_t, N_0, N)$ in the statement of 3.1, we see that $C_\lambda(\gamma, N_0, N) \leq C_{L,\lambda}(2\beta\epsilon)$, where

$$C_{L,\lambda}(\mu) = \frac{1 + \|\lambda\| - 2|\mu|_\lambda}{\|\lambda\| - 1}.$$

For any natural number n , let $C(n) = \prod_\ell C_\ell(n)$ where

$$C_\ell(n) = \frac{1 + \ell - 2|n|_\ell}{\ell - 1}.$$

Then, for any place λ over ℓ , $C_{L,\lambda}(\mu) \leq C_\ell(N_{L,\mathbb{Q}}(\mu))$. Since there are at most g primes of L lying over any rational prime ℓ , it suffices to produce a bound for $C(n)$.

One can easily show that, if $\ell_1 < \ell_2$ are primes, then $C(\ell_1) > C(\ell_2)$ and $C(\ell_1^a \ell_2^b) > \max(C(\ell_1^{a+b}), C(\ell_2^{a+b}))$. Roughly speaking, $C(n)$ is largest for n which are the product of many small primes. More precisely, let $x(n)$ be the logarithm of the n^{th} prime; then

$$C(n) < \prod_{\ell \leq x(n)} C(\ell).$$

There is a constant b so that $x(n) < b(\log n \log \log n)$. In fact, there is a constant c_0 such that $\prod_{\ell < x(n)} \frac{\ell}{\ell-1} < c_0 \log x(n)$. (Briefly, expand the product as a sum to show that $\prod_{\ell < x(n)} \frac{\ell}{\ell-1} < \sum_{i \geq 0} 2^{-i}$. (the $x(n)^{\text{th}}$ harmonic number), where $x(n)$ is as above.) Then,

$$C(n) < \prod_{\ell < x(n)} C(\ell) = \prod_{\ell < x(n)} \frac{\ell + 2}{\ell};$$

moreover,

$$\frac{C(n)}{\prod_{\ell < x(n)} \frac{\ell}{\ell-1}} = \prod_{\ell < x(n)} 1 + \frac{\ell-2}{\ell^2} < \prod_{\ell < x(n)} \frac{\ell}{\ell-1} < c_0 \log x(n).$$

Hence, there is a constant c so that

$$C(n) < c(\log \log n \log \log \log n)^2.$$

This bound on $C(n)$ produces a bound for $C_L(2\beta\epsilon) := \prod_\lambda C_{L,\lambda}(2\beta\epsilon)$, and thus for $C(\gamma, N)$, of the desired shape.

We now turn our attention to the $\Delta_w(\gamma_t)$ for w infinite, and thence to $\chi(t)$. One quickly reduces to the case where X is simple; we subsequently assume this true. Let $z \in M$ be the Frobenius π_X of X ; z^t is then the Frobenius of $X \times \text{Spec } k_t$. Every real embedding w of L extends to a pair of conjugate embeddings of $M = \text{End}_{\mathcal{O}_L}(X) \otimes \mathbb{Q}$ into \mathbb{C} . Let $\theta_w \in [0, \pi)$ denote the argument of $w(z)$ taken mod π . (For the rest of this section – in unfortunate contrast to the rest of this paper – π denotes the ratio of a circle's circumference to its diameter.) Note that θ_w/π *cannot* be rational; for if it were, then some z^t would have a real conjugate, and X would be supersingular (see [T], example (a)). Thus, $\{t\theta_w \bmod \pi\}_{t \in \mathbb{N}}$ are evenly distributed in $[0, \pi)$.

By a classical theorem of Weil, $|z|_w = |w(z)| = \sqrt{q}$. Consequently,

$$\begin{aligned} \prod_{w|v} \Delta_w(\gamma_t) &= \prod_{w|\infty} \left| \frac{(z^t - \bar{z}^t)^2}{z^t \bar{z}^t} \right|_w^{\frac{1}{2}} \\ &= \prod_{w|\infty} \left| \frac{(w(z)^t - w(\bar{z})^t)^2}{q^t} \right|^{\frac{1}{2}} \\ &= q^{-\frac{tg}{2}} \prod_{w|\infty} |2\Im(w(z^t))| \\ &= q^{-\frac{tg}{2}} \prod_{w|\infty} |\sin(t\theta_w)| 2q^{\frac{t}{2}} \\ &= 2^g \prod_{w|\infty} |\sin(t\theta_w)|. \end{aligned}$$

Thus,

$$\chi(t) = \prod_{w|\infty} |\sin(t\theta_w)|$$

is between zero and one. Depending on the structure of X , there may be relations among the θ_w for various w . Nonetheless, for each w the numbers $\{t\theta_w \bmod \pi\}_{t \in \mathbb{N}}$ are equidistributed. For any particular X , this information may be parlayed into an explicit expected value for $\chi(t)$ which depends only on the initial X . \blacksquare

Remark: Our claimed bound on $C(\gamma_t, N_0, N)$ is far from best possible. It does, in some sense, show that $|\mathcal{Y}_{(X,\iota,\alpha)}(k_t)|$ grows as $q^{\frac{gt}{2}}$.

APPENDIX: EXAMPLES

TABLE 1.

Size of the isogeny class of $E : y^2 = x^3 + x$ and $X := E \otimes \mathbb{Z}[\sqrt{7}]$ over extensions of $\mathbb{F}_q := \mathbb{F}_{13^4}$.

t	$ \mathcal{Y}_{(E,\alpha)}(\mathbb{F}_{q^t}) $	$ \mathcal{Y}_{(X,\iota,\alpha)}(\mathbb{F}_{q^t}) $
1	10560	27878400
2	3378240	8876393164800
3	634920000	249270523476480000
4	1663709760	2117325275457638400
5	9801992692800	48039530374852297585920000
6	6064454714760000	66953794524264997824176916480000
7	238088794428936000	56686274032624146070354092096000000
8	2992300423211880000	6639336978937709948209064433600000000

Without belaboring the point, we would like to mention that the formula in 3.1 truly lends itself to explicit computations. By way of evidence for this assertion, we will present here two examples.

Consider the elliptic curve with affine model $y^2 = x^3 + x$; it's actually an abelian scheme \tilde{E} over $\mathbb{Z}[\frac{1}{2}]$. It has complex multiplication by $\sqrt{-1}$, at least étale-locally on the base. Thus, \tilde{E} has ordinary reduction at p if and only if $p \equiv 1 \pmod{4}$. Let $E_0 = \tilde{E} \times_{\text{Spec } \mathbb{Z}[\frac{1}{2}]} \text{Spec } \mathbb{F}_{13}$ be the reduction of E at $p = 13$.

We may choose an identification of $\text{End}(E_0)$ with $\mathbb{Z}[\sqrt{-1}]$ so that the Frobenius of E is given by $-3 + 2\sqrt{-1}$. We want a $\Gamma_0(N_0, N)$ -level structure on E with N squarefree and strictly larger than 2. This doesn't happen over \mathbb{F}_p , or even over \mathbb{F}_{p^2} ; but over \mathbb{F}_{p^4} , the elliptic curve acquires full 3- and 5-torsion. So let $q = p^4$; there is a $\Gamma_0(5, 3)$ -structure α on $E := E_0 \times_{\text{Spec } \mathbb{F}_p} \text{Spec } \mathbb{F}_q$. Evaluation of the terms in 3.1 yields the middle column of Table 1.

To an elliptic curve and an order in a totally real field one can canonically associate a Hilbert-Blumenthal abelian variety. We will carry out this procedure to produce our second example. Let L be the totally real field $\mathbb{Q}(\sqrt{7})$, with ring of integers $\mathcal{O}_L = \mathbb{Z}[\sqrt{7}]$. Let $\tilde{X} = \tilde{E} \otimes_{\mathcal{O}_L}$; it is an abelian surface over $\text{Spec } \mathbb{Z}[\frac{1}{2}]$ with a canonical action $\iota : \mathcal{O}_L \rightarrow \text{End}(\tilde{X})$. The level structure α on \tilde{E} similarly induces a level structure β on \tilde{X} . Let

$X = \tilde{X} \times \text{Spec } \mathbb{F}_q$. The last column in Table 1 records the size of the isogeny class of (X, ι, α) in the moduli space of abelian surfaces with $\Gamma_0(5, 3)$ -level structure and an action of $\mathbb{Z}[\sqrt{7}]$.

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