Homework 9
Due: Friday, April 10

1. [Katz] Let $N = 2n+1$, and consider $\mathbb{P}^N$ with coordinates $X_0, \cdots, X_n, Y_0, \cdots, Y_n$. Let
$$F = \sum_{i=0}^{n} X_i Y_i^q - X_i^q Y_i,$$
and let $Z/\mathbb{F}_q$ be the (smooth, irreducible, projective) hypersurface
$$Z = \mathcal{Z}_{\mathbb{P}^N_{\mathbb{F}_q}}(F).$$
Show that for every hyperplane $\mathcal{Z}(L) \subset \mathbb{P}^N_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, $\mathcal{Z}(L) \cap Z$ is not smooth.

   If you like, you may proceed as follows.

   (a) Show that $Z(\mathbb{F}_q) = \mathbb{P}^N(\mathbb{F}_q)$, i.e., that every point $[a_0, \cdots, a_n, b_0, \cdots, b_n]$ with each $a_i, b_j \in \mathbb{F}_q$ lies on Z. (Such a point corresponds to a closed point P of the scheme Z.)
   (b) Suppose that $a_0 = 1$, and let f be the dehomogenization $f = f(1, x_1, \cdots, x_n, y_0, \cdots, y_n)$. Compute the linearization $d_P(f)$. (HINT: *See 672HW7#5.*)
   (c) Let $\ell \in \mathbb{F}_q[x_1, \cdots, x_n, y_1, \cdots, y_n]$ be a linear polynomial. Show that there exists $P \in Z$ as above such that $T_P(Z) = \mathcal{Z}_{\mathbb{A}^N}(\ell)$.
   (d) Finish the claim.

2. Calculate the zeta function of $\mathbb{P}^1_{\mathbb{F}_q}$.

3. Let $S = \operatorname{Spec} \mathbb{F}_q$.

   (a) Let $P = \operatorname{Spec} \mathbb{F}_{q^d}$. Calculate
$$\# \operatorname{Mor}_S(\operatorname{Spec} \mathbb{F}_{q^e}, P).$$
   (HINT: *Try the case $d = e$ first.*)
   (b) Suppose X/S is a scheme of finite type. Show that
$$\#X(\mathbb{F}_{q^e}) = \sum_{d|e} d X_d, \tag{1}$$
   where X_d is the set of closed points of X of degree d. (The hypothesis "finite type" is just to ensure that all quantities in (1) are finite.

4. Let $X/\mathbb{F}_q$ be a scheme of finite type. Show that there is an equality of generating functions
$$\prod_{P \in X \text{ closed}} (1 - T^{\deg P}) = \exp\Big(\sum_{r \geq 1} \frac{\#X(\mathbb{F}_{q^r})}{r} T^r\Big).$$
(HINT: *Take logarithms...*)

5. (a) Suppose $f(x) \in \mathbb{F}_q[x]$ is nonconstant. Show that $\mathcal{Z}_{\mathbb{A}^1}(f)$ is smooth if and only if $f(x)$ is squarefree.

   (b) Using Poonen's theorem, compute

$$\lim_{d\to\infty} \frac{\#\{f(x) \in \mathbb{F}_q[x] : \deg f \le d, f \text{ squarefree}\}}{\#\{f(x) \in \mathbb{F}_q[x] : \deg f \le d\}}.$$