Homework 7

Due: Friday, March 13

Let X be a scheme and $P \in X$. Let $\mathfrak{m}_P$ be the maximal ideal of $\mathcal{O}_{X,P}$ and $\kappa(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$ be the residue field. (In the case where $X = \operatorname{Spec} A$ and $P = [\mathfrak{p}]$, $\kappa(P) = \operatorname{Frac} A/\mathfrak{p}$.)

The cotangent space to X at P is $(T_P X)^\vee = \mathfrak{m}_P/\mathfrak{m}_P^2 \otimes_{\mathcal{O}_{X,P}} \kappa(P)$; the tangent space is $T_P X = (\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee = \operatorname{Hom}((T_P X)^\vee, \kappa(P))$.

If $f : X \to Y$ is a morphism of schemes and $P = f(Q)$, then f induces maps $\mathfrak{n}_Q \to \mathfrak{m}_P$, and thus maps $\kappa(Q) \hookrightarrow \kappa(P)$, $(T_P Y)^\vee \to (T_P X)^\vee$ and a $\kappa(P)$-linear map

$$T_P X \xrightarrow{d_P f} T_Q Y \otimes_{\kappa(Q)} \kappa(P).$$

1. Consider the closed subscheme $X = \operatorname{Spec} k[x,y]/(y^2 - (x^3 - x)) = \operatorname{Spec} A$ of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y] = \operatorname{Spec} B$, equipped with its natural inclusion $\iota : X \to \mathbb{A}^2_k$.

   Let P be the origin. Viewed as a point of X, it corresponds to a maximal ideal $\mathfrak{m} = \mathfrak{m}_P \subset A$; as a point of $\mathbb{A}^2_k$, it corresponds to a maximal ideal $\mathfrak{n} = \mathfrak{n}_P \subset B$.

   (a) Calculate $(T_P X)^\vee$, $T_P X$, $(T_P Y)^\vee$ and $T_P Y$.

   (b) Calculate the induced maps $(T_P Y)^\vee \to (T_P X)^\vee$ and $d_P \iota : T_P X \to T_P Y$. *There's a canonical isomorphism of residue fields $A/\mathfrak{m}_P \cong B/\mathfrak{n}_P \cong k$, so no need to worry about residue fields.*

   (c) [VQA] Draw a picture which explains all this.

2. Let $X = \operatorname{Spec} \mathbb{Z}[i]$, $Y = \operatorname{Spec} \mathbb{Z}$, and $f : X \to Y$ be the natural morphism of schemes.

   (a) Let $P = [(1+2i)\mathbb{Z}[i]] \in X$ and $Q = [5\mathbb{Z}] \in Y$. Compute $d_P f$.

   (b) Same thing for $P = [3\mathbb{Z}[i]]$, $Q = [3\mathbb{Z}]$.

   (c) Same thing for $P = [(1+i)\mathbb{Z}[i]]$, $Q = [2\mathbb{Z}]$.

3. Let $(A, \mathfrak{m})$ be a Noetherian local ring. The formal completion of A at $\mathfrak{m}$, or the $\mathfrak{m}$-adic completion of A, is

$$\widehat{A} = \varprojlim_n A/\mathfrak{m}^n.$$

   It is a local ring, with maximal ideal $\widehat{\mathfrak{m}} = \varprojlim_n \mathfrak{m}/\mathfrak{m}^n$.

   (a) Describe the natural map

$$A \longrightarrow \widehat{A}.$$

   Show that it is an injection. You may use the following result, stated last semester, which follows from the Artin-Rees Theorem:

Theorem Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then $\cap_{n \geq 0} \mathfrak{m}^n = (0)$.

(b) Calculate the completion for the following pairs:

i. $(k[x]_{(x)}, (x))$;

ii. $(\mathbb{Z}_{(p)}, (p))$.

(c) Let A be a ring and $\mathfrak{p} \subset A$ prime. Show that

$$\varprojlim_n A/\mathfrak{p}^n \cong \widehat{A_{\mathfrak{p}}}.$$

(HINT: *Localization is exact.*)

4. Let $X = \operatorname{Spec} A$ be an irreducible affine scheme. Suppose $P \in X$.

(a) There is a canonical map between X and $\operatorname{Spec} \widehat{\mathcal{O}}_{X,P}$. Which way does it go?

(b) Suppose $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$, $P = (t)$. Describe the closure of the image of this morphism.

The answer may be surprising. The formal completion of X at P is the ringed space in which the underlying space is the point P and the structure sheaf is $\widehat{\mathcal{O}}_{X,P}$. Such a ringed space is an example of a formal scheme; in general, it's not actually a scheme.

5. Let k be a field in which two is invertible. Let $A = k[x,y]/(y^2 - x^2(x+1))$, and let $X = \operatorname{Spec} A$. Let $\mathfrak{p} = (x-1, y-1)A$, $\mathfrak{q} = (x,y)$, $P = [\mathfrak{p}]$ and $Q = [\mathfrak{q}]$.

(a) Calculate the tangent spaces $T_P X$ and $T_Q X$.

(b) Show that $\widehat{\mathcal{O}}_{X,P} \cong k[[t]]$.

Bonus: Show that $\widehat{\mathcal{O}}_{X,Q} \cong k[[tu]]/(tu)$.