Let $X$ be a scheme and $P \in X$. Let $m_P$ be the maximal ideal of $O_{X,P}$ and $\kappa(P) = O_{X,P}/m_P$ be the residue field. (In the case where $X = \text{Spec } A$ and $P = [p]$, $\kappa(P) = \text{Frac } A/p$.)

The cotangent space to $X$ at $P$ is $(T_P X)'^\vee = m_P/m_P^2 \otimes_{O_{X,P}} \kappa(P)$; the tangent space is $T_P X = (m_P/m_P^2)^\vee = \text{Hom}((T_P X)'^\vee, \kappa(P))$.

If $f : X \to Y$ is a morphism of schemes and $P = f(Q)$, then $f$ induces maps $n_Q \to m_P$, and thus maps $\kappa(Q) \to \kappa(P)$, $(T_P Y)' \to (T_P X)'$ and a $\kappa(P)$-linear map

$$T_P X \xrightarrow{d_P f} T_Q Y \otimes_{\kappa(Q)} \kappa(P).$$

1. Consider the closed subscheme $X = \text{Spec } k[x, y]/(y^2 - (x^3 - x)) = \text{Spec } A = \mathbb{A}_k^2 = \text{Spec } B$, equipped with its natural inclusion $i : X \to \mathbb{A}_k^2$.

   Let $P$ be the origin. Viewed as a point of $X$, it corresponds to a maximal ideal $m = m_P \subset A$; as a point of $\mathbb{A}_k^2$, it corresponds to a maximal ideal $n = n_P \subset B$.

   (a) Calculate $(T_P X)'$, $T_P X$, $(T_P Y)'$ and $T_P Y$.

   (b) Calculate the induced maps $(T_P Y)' \to (T_P X)'$ and $d_P : T_P X \to T_P Y$. There’s a canonical isomorphism of residue fields $A/m_P \cong B/n_P \cong k$, so no need to worry about residue fields.

   (c) [VQA] Draw a picture which explains all this.

2. Let $X = \text{Spec } \mathbb{Z}[i]$, $Y = \text{Spec } \mathbb{Z}$, and $f : X \to Y$ be the natural morphism of schemes.

   (a) Let $P = [(1 + 2i)\mathbb{Z}[i]] \in X$ and $Q = [5\mathbb{Z}] \in Y$. Compute $d_P f$.

   (b) Same thing for $P = [3\mathbb{Z}[i]]$, $Q = [3\mathbb{Z}]$.

   (c) Same thing for $P = [(1 + i)\mathbb{Z}[i]]$, $Q = [2\mathbb{Z}]$.

3. Let $(A, m)$ be a Noetherian local ring. The formal completion of $A$ at $m$, or the $m$-adic completion of $A$, is

   $\hat{A} = \lim_{\leftarrow n} A/m^n$.

   It is a local ring, with maximal ideal $\hat{m} = \lim_{\leftarrow n} m/m^n$.

   (a) Describe the natural map

   $A \longrightarrow \hat{A}$.

   Show that it is an injection. You may use the following result, stated last semester, which follows from the Artin-Rees Theorem:
**Theorem**  Let \((A, m)\) be a Noetherian local ring. Then \(\bigcap_{n \geq 0} m^n = (0)\).

(b) Calculate the completion for the following pairs:
   
   i. \((k[x]_{(x)}, (x))\);
   
   ii. \((\mathbb{Z}(p), (p))\).

(c) Let \(A\) be a ring and \(p \subset A\) prime. Show that

\[
\lim_{\rightarrow n} A/p^n \cong \hat{A}_p.
\]

(HINT: Localization is exact.)

4. Let \(X = \text{Spec } A\) be an irreducible affine scheme. Suppose \(P \in X\).

   (a) There is a canonical map between \(X\) and \(\text{Spec } \hat{O}_{X,P}\). Which way does it go?

   (b) Suppose \(X = \mathbb{A}^1 = \text{Spec } k[t], P = (t)\). Describe the closure of the image of this morphism.

   *The answer may be surprising. The formal completion of \(X\) at \(P\) is the ringed space in which the underlying space is the point \(P\) and the structure sheaf is \(\hat{O}_{X,P}\). Such a ringed space is an example of a formal scheme; in general, it’s not actually a scheme.*

5. Let \(k\) be a field in which two is invertible. Let \(A = k[x,y]/(y^2 - x^2(x+1))\), and let \(X = \text{Spec } A\). Let \(p = (x-1, y-1)A, q = (x, y), P = [p]\) and \(Q = [q]\).

   (a) Calculate the tangent spaces \(T_P X\) and \(T_Q X\).

   (b) Show that \(\hat{O}_{X,P} \cong k[t]\).

   **Bonus:** Show that \(\hat{O}_{X,Q} \cong k[[t]]/(tu)\).