Homework 6

Due: Friday, March 6

1. [*Generic points*] Suppose $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of schemes, coming from $\phi : B \to A$. It is not hard to show that the closure of the image of f is

$$\overline{\operatorname{im}(f)} = \mathcal{Z}(\ker \phi).$$

(Use, e.g., HW2#1.)

Recall that $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$.

   (a) Describe all points P of $\mathbb{A}^1_{\mathbb{Q}}$ such that $\overline{P} = \mathbb{A}^1$.

   (b) In each of the following examples, describe the closure of the image of the morphism.

      i. $\operatorname{Spec} \mathbb{Q} \hookrightarrow \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$ given by $x \mapsto 1$.

      ii. $\operatorname{Spec} \mathbb{Q}(i) \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$ given by $x \mapsto i$.

      iii. $\operatorname{Spec} \mathbb{Q}(\pi) \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$ given by $x \mapsto \pi$.

   (c) Let K be a field of characteristic zero. Describe alll morphisms (if any) $\operatorname{Spec} K \to \mathbb{A}^1_{\mathbb{Q}}$ with dense image.

2. Use the valuative criterion to show that $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper.

A morphism $f : X \to Y$ of schemes is *finite* if for every affine open subset $V \subset Y$, $f^{-1}(V)$ is affine and $\mathcal{O}_X(f^{-1}(V))$ is a finite module over $\mathcal{O}_Y(V)$.

Equivalently [Liu ex. 3.3.15] there exists some open affine cover $Y = \cup Y_i$ such that $f^{-1}(Y_i)$ is affine and $\mathcal{O}_X(f^{-1}(Y_i))$ is finite over $\mathcal{O}_Y(Y_i)$.

A morphism is *quasifinite* if, for each $Q \in Y$, $f^{-1}(Q)$ is a finite set.

3. (a) Show that any finite morphism is quasifinite.

   (b) Show that the converse is false.(HINT: *672HW3#1b*).

4. (a) Show that the property of being quasifinite is stable under base change.

   (b) Show that the property of being finite is stable under base change.

5. Consider a finite morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$. A version of the going-up theorem says:

Theorem Suppose that $B \subset A$ is an integral extension of rings. Suppose $\mathfrak{q} \subset B$ is prime; then there exists a prime $\mathfrak{p}$ of A such that $\mathfrak{p} \cap B = \mathfrak{q}$. Moreover, $\mathfrak{p}$ may be chosen to contain any given ideal I which satisfies the (obvious necessary) condition $I \cap B \subseteq \mathfrak{q}$.

(This formulation comes from Eisenbud's *Commutative algebra*, Prop. 4.15; see also 672HW10#7.)

(a) Suppose that f induces an inclusion $B \subset A$; show that f is a closed map.
(b) Continue to suppose $B \subset A$; show that f is proper. (HINT: *4.(b)*)

Problems 4 and 5 can be used to assemble a proof of:

Theorem A finite morphism is proper.