Homework 5
Due: February 27

If Y is a scheme and $Q \in Y$, there is a natural inclusion $\iota_Q : Q \hookrightarrow Y$. If $X \xrightarrow{\alpha} Y$ is a morphism of schemes, the *fiber of X over Q is*

$$\begin{tikzcd} X_Q = Q \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ Q \arrow[r, hook, "\iota_Q"] & Y \end{tikzcd}$$

1. Let $X = \operatorname{Spec} \mathbb{Z}[x]/(x^2+1)$ and $Y = \operatorname{Spec} \mathbb{Z}$; consider the map $X \to Y$ which comes from $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]/(x^2+1)$.

   Suppose $Q \in Y$. Describe the fiber X_Q. (HINT: *Your answer will depend on* $Q = [(q)]$.)

2. Let X and Y be S-schemes. Use the universal property of fiber products to show that there is a natural bijection of sets

$$(X \times_S Y)(T) \longrightarrow X(T) \times_{S(T)} Y(T).$$

   (HINT: *Compare Liu, Example 3.1.6, page 81.*)

3. (a) Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.

   (b) Let $X = \operatorname{Spec} k[x]$, $Y = \operatorname{Spec} k[y]$, and let ϕ be the morphism $X \to Y$ attached to

$$\begin{array}{ccc} k[y] & \longrightarrow & k[x] \\ y & \longrightarrow & x^2 \end{array}$$

   i. Suppose char $(k) \neq 2$. Show that $X \times_Y X$ has two irreducible components.

   ii. Suppose char $(k) = 2$. Describe $X \times_Y X$.

   (HINT: *It may be marginally easier to think of two copies of X, X_1 and X_2, with coordinates x_1 and x_2...*)