Homework 4  
Due: Friday, February 20  

Even though the homework isn’t due until February 20, please do problem 1 before class on Monday, February 16.

1. Suppose $S$ and $T$ and $U$ are sets and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of $S$ and $T$ over $U$, denoted $S \times_{f \circ U \circ f} T$ or $S \times_U T$ if the maps are understood, is

$$S \times_U T := \{(s, t) : s \in S, t \in T, f(s) = g(t)\}.$$

This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it’s the smallest set which makes the following diagram commute:

(a) Consider the following sets:

- $S = \{1, 2, 3, 4\}$
- $T = \{a, b, c, d, e, f\}$
- $U = \{\alpha, \beta\}$
- $V = \{\gamma\}$

and the following maps between them:

$$S \xrightarrow{f} U \quad T \xrightarrow{g} U$$

$$1, 2 \quad \alpha \quad a, b, c \quad \alpha$$

$$3, 4 \quad \beta \quad d, e, f \quad \beta$$

$$S \xrightarrow{p} V \quad T \xrightarrow{q} V$$

anything $\quad \gamma \quad$ anything $\quad \gamma$

What is $S \times_U T$? What is $S \times_V T$?
(b) Suppose \( S \) and \( T \) are arbitrary sets, \( U \) is a set with one element, and \( f : S \to U \) and \( g : T \to U \) are the maps which send everything to the (unique) element of \( U \). What is \( S \times_U T \)?

(c) Suppose \( S \) and \( T \) are subsets of \( U \); denote the inclusions by \( i : S \hookrightarrow U \) and \( j : T \hookrightarrow U \). What is \( S \times_{i,U,j} T \)?

2. Let \( X \) be the affine scheme
\[
X = \text{Spec } \frac{\mathbb{Z}[x,y,z]}{(x^2 + y^2 - z^2)}.
\]
Let \( A \) be any ring. Carefully explain the bijection between:

- \( X(A) \), i.e., maps of schemes \( \text{Spec } A \to X \); and
- Triples \( \alpha, \beta, \gamma \in A \) such that \( \alpha^2 + \beta^2 = \gamma^2 \).

3. [VQA]

(a) Describe the relationship between the sets \( \mathbb{A}^1_\mathbb{C} \) and \( \mathbb{A}^1_\mathbb{C}(\mathbb{C}) \).

(b) Let \( K = \mathbb{Q}(\sqrt{-1}) \). Describe the relationship between the sets \( \mathbb{A}^1_\mathbb{Q} \) and \( \mathbb{A}^1_\mathbb{Q}(K) \).

4. Let \( A \) and \( B \) be rings, and consider the ring \( A \oplus B \). Note that the natural surjection \( A \oplus B \to A \) corresponds to a closed immersion \( \text{Spec } A \hookrightarrow \text{Spec } (A \oplus B) \).

Show that, as subscheme of \( \text{Spec } (A \oplus B) \), \( \text{Spec } A \) is both open and closed. (HINT: Consider \( D((1,0)) \).)

5. Let \( A \) be a ring. An element \( e \in A \) is called a nontrivial idempotent if \( e^2 = e \) but \( e \not\in \{0,1\} \).

For example, if \( B \) and \( C \) are rings, then the element \( (1,0) \in B \oplus C \) is a nontrivial idempotent of \( B \oplus C \).

(a) List all the idempotents of the ring \( \mathbb{C}[x]/(x^2 - x) \).

(b) Let \( e \) be an idempotent of \( A \). Show that the map
\[
\begin{array}{ccc}
A & \longrightarrow & eA \oplus (1 - e)A \\
a & \longmapsto & (ea, (1 - e)a)
\end{array}
\]
is an isomorphism.

(c) Show that \( \text{Spec } A \) is not connected if and only if \( A \) has a nontrivial idempotent. (HINT: \( \text{Suppose } \text{Spec } A = Z(I) \cup Z(J) \) with \( Z(I) \cap Z(J) = \emptyset \); show \( \text{Spec } A \) is homeomorphic to \( \text{Spec } (A/I \oplus A/J) \).)