Homework 4
Due: Friday, February 20

Even though the homework isn't due until February 20, please do problem 1 before class on Monday, February 16.

1. Suppose S and T and U are *sets* and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of S and T over U, denoted $S \times_{f,U,g} T$ or $S \times_U T$ f the maps are understood, is
$$S \times_U T := \{(s,t) : s \in S, t \in T, f(s) = g(t)\}.$$
This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it's the smallest set which makes the following diagram commute:
$$\begin{tikzcd} S \times_U T \arrow[r] \arrow[d] & T \arrow[d, "g"] \\ S \arrow[r, "f"] & U \end{tikzcd}$$

   (a) Consider the following sets:
$$\begin{aligned} S &= \{1,2,3,4\} \\ T &= \{a,b,c,d,e,f\} \\ U &= \{\alpha,\beta\} \\ V &= \{\gamma\} \end{aligned}$$
   and the following maps between them:
$$S \xrightarrow{f} U \qquad T \xrightarrow{g} U$$
$$1,2 \longmapsto \alpha \qquad a,b,c \longmapsto \alpha$$
$$3,4 \longmapsto \beta \qquad d,e,fg \longmapsto \beta$$
$$S \xrightarrow{p} V \qquad T \xrightarrow{q} V$$
$$\text{anything} \longmapsto \gamma \quad \text{anything} \longmapsto \gamma$$
   What is $S \times_U T$? What is $S \times_V T$?

(b) Suppose S and T are arbitrary sets, U is a set with one element, and $f : S \to U$ and $g : T \to U$ are the maps which send everything to the (unique) element of U. What is $S \times_U T$?

(c) Suppose S and T are subsets of U; denote the inclusions by $i : S \hookrightarrow U$ and $j : T \hookrightarrow U$. What is $S \times_{i,U,j} T$?

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2. Let X be the affine scheme
$$X = \operatorname{Spec} \frac{\mathbb{Z}[x,y,z]}{(x^2+y^2-z^2)}.$$
Let A be any ring. Carefully explain the bijection between:
   - $X(A)$, i.e., maps of schemes $\operatorname{Spec} A \to X$; and
   - Triples $\alpha, \beta, \gamma \in A$ such that $\alpha^2 + \beta^2 = \gamma^2$.

3. [VQA]
   (a) Describe the relationship between the sets $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$.
   (b) Let $K = \mathbb{Q}(\sqrt{-1})$. Describe the relationship between the sets $\mathbb{A}^1_{\mathbb{Q}}$ and $\mathbb{A}^1_{\mathbb{Q}}(K)$.

4. Let A and B be rings, and consider the ring $A \oplus B$. Note that the natural surjection $A \oplus B \to A$ corresponds to a closed immersion $\operatorname{Spec} A \hookrightarrow \operatorname{Spec}(A \oplus B)$.

   Show that, as subscheme of $\operatorname{Spec}(A \oplus B)$, $\operatorname{Spec} A$ is both open and closed. (HINT: *Consider* $D((1,0))$.)

5. Let A be a ring. An element $e \in A$ is called a nontrivial idempotent if $e^2 = e$ but $e \notin \{0,1\}$.

   For example, if B and C are rings, then the element $(1,0) \in B \oplus C$ is a nontrivial idempotent of $B \oplus C$.

   (a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2-x)$.
   (b) Let e be an idempotent of A. Show that the map
$$\begin{aligned} A &\longrightarrow eA \oplus (1-e)A \\ a &\longmapsto (ea, (1-e)a) \end{aligned}$$
   is an isomorphism.
   (c) Show that $\operatorname{Spec} A$ is not connected if and only if A has a nontrivial idempotent. (HINT: *Suppose* $\operatorname{Spec} A = Z(I) \cup Z(J)$ *with* $Z(I) \cap Z(J) = \emptyset$*; show* $\operatorname{Spec}(A)$ *is homeomorphic to* $\operatorname{Spec}(A/I \oplus A/J)$.)