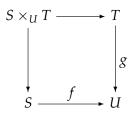
Homework 4 Due: Friday, February 20

Even though the homework isn't due until February 20, please do problem 1 before class on Monday, February 16.

1. Suppose *S* and *T* and *U* are *sets* and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of *S* and *T* over *U*, denoted $S \times_{f,U,g} T$ or $S \times_U T$ f the maps are understood, is

$$S \times_{U} T := \{(s, t) : s \in S, t \in T, f(s) = g(t)\}$$

This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it's the smallest set which makes the following diagram commute:



(a) Consider the following sets:

$$S = \{1, 2, 3, 4\}$$

$$T = \{a, b, c, d, e, f\}$$

$$U = \{\alpha, \beta\}$$

$$V = \{\gamma\}$$

and the following maps between them:

- $S \xrightarrow{f} U \qquad T \xrightarrow{g} U$ $1, 2 \longmapsto \alpha \qquad a, b, c \longmapsto \alpha$ $3, 4 \longmapsto \beta \qquad d, e, fg \longmapsto \beta$
- $S \xrightarrow{p} V \qquad T \xrightarrow{q} V$
- anything $\longmapsto \gamma$ anything $\longmapsto \gamma$

What is $S \times_U T$? What is $S \times_V T$?

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- (b) Suppose *S* and *T* are arbitrary sets, *U* is a set with one element, and *f* : *S* → *U* and *g* : *T* → *U* are the maps which send everything to the (unique) element of *U*. What is *S* ×_{*U*} *T*?
- (c) Suppose *S* and *T* are subsets of *U*; denote the inclusions by $i : S \hookrightarrow U$ and $j : T \hookrightarrow U$. What is $S \times_{i,U,j} T$?
- 2. Let *X* be the affine scheme

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x, y, z]}{(x^2 + y^2 - z^2)}.$$

Let *A* be any ring. Carefully explain the bijection between:

- X(A), i.e., maps of schemes Spec $A \rightarrow X$; and
- Triples α , β , $\gamma \in A$ such that $\alpha^2 + \beta^2 = \gamma^2$.
- 3. [VQA]
 - (a) Describe the relationship between the sets $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$.
 - (b) Let $K = \mathbb{Q}(\sqrt{-1})$. Describe the relationship between the sets $\mathbb{A}^1_{\mathbb{Q}}$ and $\mathbb{A}^1_{\mathbb{Q}}(K)$.
- 4. Let *A* and *B* be rings, and consider the ring *A* ⊕ *B*. Note that the natural surjection *A* ⊕ *B* → *A* corresponds to a closed immersion Spec *A* → Spec(*A* ⊕ *B*).
 Show that, as subscheme of Spec(*A* ⊕ *B*), Spec *A* is both open and closed. (HINT: *Consider D*((1,0)).)
- 5. Let *A* be a ring. An element $e \in A$ is called a nontrivial idempotent if $e^2 = e$ but $e \notin \{0, 1\}$. For example, if *B* and *C* are rings, then the element $(1, 0) \in B \oplus C$ is a nontrivial idempotent of $B \oplus C$.
 - (a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2 x)$.
 - (b) Let *e* be an idempotent of *A*. Show that the map

$$A \longrightarrow eA \oplus (1-e)A$$
$$a \longmapsto (ea, (1-e)a)$$

is an isomorphism.

(c) Show that Spec *A* is not connected if and only if *A* has a nontrivial idempotent. (HINT: *Suppose* Spec $A = Z(I) \cup Z(J)$ with $Z(I) \cap Z(J) = \emptyset$; show Spec(*A*) is homeomorphic to Spec($A/I \oplus A/J$).)

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