Homework 3
Due: Friday, February 13

1. Let R be a ring. The nilradical of R is the set of all nilpotent elements of R:

$$\mathcal{N}(R) = \{a \in R : \exists n \in \mathbb{N} : a^n = 0\}.$$

   (a) Show that
$$\mathcal{N}(R) = \cap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}.$$

   (b) Show that $R/\mathcal{N}(R)$ is reduced, i.e., that $\mathcal{N}(R/\mathcal{N}(R)) = (0)$.

2. (a) Show that $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R/\mathcal{N}(R))$ are homeomorphic topological spaces.

   (b) When are $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R/\mathcal{N}(R))$ isomorphic schemes?

   (c) Suppose $p, q \in \mathbb{Z}$ are distinct primes, $m, n \in \mathbb{N}$. Show that as a topological space, $\operatorname{Spec}(\mathbb{Z}/p^m q^n)$ is a two-point space with the discrete topology.

3. (a) Let R be an integral domain and let $X = \operatorname{Spec} R$. Let $\eta = [(0)] \in X$. Use the definition of a stalk to compute $\mathcal{O}_{X,\eta}$.

   (b) [*VQA**] Relate this to last semester's field of rational functions on an irreducible variety.

4. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and let $U = \mathbb{A}^2 - \{[(x, y)]\}$.
   Let $V_1 = D(x)$ and $V_2 = D(y)$.

   (a) Verify that $U = V_1 \cup V_2$ is an open cover of U.

   (b) Calculate $\mathcal{O}_X(V_1)$, $\mathcal{O}_X(V_2)$ and $\mathcal{O}_X(V_1 \cap V_2)$.

   (c) Use this to calculate $\mathcal{O}_X(U)$.

   So, if U were affine, we would have $U = \operatorname{Spec}(\mathcal{O}_X(U))$; but this isn't true!

5. Suppose $Y = \operatorname{Spec}(R/I)$ and $Z = \operatorname{Spec}(R/J)$ are closed subschemes of $X = \operatorname{Spec}(R)$.
   Say that Y contains Z if Z is a closed subscheme of Y, i.e., if $J \supset I$.
   The (scheme-theoretic) union and intersection of Y and Z are:

   - $Y \cup Z = \operatorname{Spec}(R/(I \cap J))$;
   - $Y \cap Z = \operatorname{Spec}(R/(I + J))$.

   Consider the following closed subschemes of $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$:

   - $Z_1 = \operatorname{Spec} k[x, y]/(x)$;

*Vague question alert! Sometimes I just want everybody to think about something, even if there's no real associated well-defined, finite question.

- $Z_2 = \operatorname{Spec} k[x, y]/(y)$;
- $Z_3 = \operatorname{Spec} k[x, y]/(y - x^2)$.

(a) Graph all Z_i.

(b) For each pair $1 \leq i < j \leq 3$, compute $Z_{ij} = Z_i \cap Z_j$ and $V_{ij} = Z_i \cup Z_j$. Graph them.

(c) For which pairs of pairs does one have $Z_{ij} \subset Z_{i'j'}$?