Homework 2
Due: Tuesday, February 3

Be prepared to present on February 3; writeup due Wednesday, February 4.

1. Liu 2.3.3. Let Spec $R$ be an affine scheme, and $U \subset \text{Spec } R$ be a subset.
   (a) Suppose $f \in R$. Show that $Z(f)$ contains $U$ if and only if, for each $[p] \in U$, $f \in p$.
   (b) Show that the closure $\overline{U}$ of $U$ in Spec $R$ is $Z(I)$, where $I = \cap_{p \in U} p$.

2. Let $\phi : R \to S$ be a ring homomorphism.
   (a) Suppose $R$ and $S$ are finitely generated $k$-algebras, where $k$ is an algebraically closed field. Show that if $Q \in \text{Spec } S$ is a closed point, then $\text{Spec}(\phi)(Q)$ is a closed point of $\text{Spec } R$.
   (b) Give an example where $Q \in \text{Spec } S$ is closed, but $\text{Spec}(\phi)(Q)$ is not closed.
   (HINT: See [HW1#3 from last semester].)

3. Let $X$ be a set with two elements, equipped with the discrete topology; so, the open subsets of $X$ are $X$; two one-point sets $U$ and $V$; and $\emptyset$.
   Consider the following presheaf $\mathcal{F}$ of abelian groups on $X$:
   \[
   \mathcal{F}(X) = \mathbb{Z} \oplus \mathbb{Z}
   \]
   \[
   \mathcal{F}(U) = \mathbb{Z}/3
   \]
   \[
   \mathcal{F}(V) = \mathbb{Z}/3
   \]
   \[
   \mathcal{F}(\emptyset) = \{e\}
   \]

   with restriction maps $\text{res}_{X,U}(a,b) = a \mod 3$ and $\text{res}_{X,V}(a,b) = b \mod 3$.
   Then $\mathcal{F}$ is not a sheaf. Why?

4. Skyscraper sheaves Let $X$ be a topological space. Let $P \in X$ be a point whose closure is $Z$.
   (a) Suppose $Q \in Z$. Let $U$ be an open neighborhood of $Q$. Show that $P \in U$.
   (b) Let $A$ be a finite abelian group. Define a sheaf $\mathcal{S}$ as follows:
   \[
   \mathcal{S}(U) = \begin{cases} A & P \in U \\ \{0\} & P \notin U \end{cases}
   \]
   Compute the stalks of $\mathcal{S}$: show that
   \[
   \mathcal{S}_Q = \begin{cases} A & Q \in Z \\ \{0\} & \text{otherwise} \end{cases}
   \]