Homework 2

Due: Tuesday, February 3

Be prepared to present on February 3; writeup due Wednesday, February 4.

1. *Liu 2.3.3.* Let $\operatorname{Spec} R$ be an affine scheme, and $U \subset \operatorname{Spec} R$ be a subset.
   (a) Suppose $f \in R$. Show that $\mathcal{Z}(f)$ contains U if and only if, for each $[\mathfrak{p}] \in U$, $f \in \mathfrak{p}$.
   (b) Show that the closure $\overline{U}$ of U in $\operatorname{Spec} R$ is $\mathcal{Z}(I)$, where $I = \cap_{\mathfrak{p} \in U} \mathfrak{p}$.
2. Let $\phi : R \to S$ be a ring homomorphism.
   (a) Suppose R and S are finitely generated k-algebras, where k is an algebraically closed field. Show that if $Q \in \operatorname{Spec} S$ is a closed point, then $\operatorname{Spec}(\phi)(Q)$ is a closed point of $\operatorname{Spec} R$.
   (b) Give an example where $Q \in \operatorname{Spec} S$ is closed, but $\operatorname{Spec}(\phi)(Q)$ is *not* closed.

   (HINT: *See HW1#3 from last semester.*)
3. Let X be a set with two elements, equipped with the discrete topology; so, the open subsets of X are X; two one-point sets U and V; and $\emptyset$.

   Consider the following presheaf $\mathcal{F}$ of abelian groups on X:

$$\begin{aligned} \mathcal{F}(X) &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathcal{F}(U) &= \mathbb{Z}/3 \\ \mathcal{F}(V) &= \mathbb{Z}/3 \\ \mathcal{F}(\emptyset) &= \{e\} \end{aligned}$$

   with restriction maps $\operatorname{res}_{X,U}(a,b) = a \bmod 3$ and $\operatorname{res}_{X,V}(a,b) = b \bmod 3$.

   Then $\mathcal{F}$ is not a sheaf. Why?
4. *Skyscraper sheaves* Let X be a topological space. Let $P \in X$ be a point whose closure is Z.
   (a) Suppose $Q \in Z$. Let U be an open neighborhood of Q. Show that $P \in U$.
   (b) Let A be a finite abelian group. Define a sheaf $\mathcal{S}$ as follows:

$$\mathcal{S}(U) = \begin{cases} A & P \in U \\ \{0\} & P \notin U \end{cases}$$

   Compute the stalks of $\mathcal{S}$: show that

$$\mathcal{S}_Q = \begin{cases} A & Q \in Z \\ \{0\} & \text{otherwise} \end{cases}.$$