Homework 1

Due: Friday, January 30

1. Describe all points of $\operatorname{Spec} \mathbb{Q}[x]$.
2. (a) Let $f = x^3 - 8 \in \mathbb{Z}[x]$. For each point $[\mathfrak{p}] \in \operatorname{Spec} \mathbb{Z}[x]$ listed below, compute the residue field $\kappa([\mathfrak{p}])$, and evaluate $f([\mathfrak{p}])$.
   i. $[(3)]$
   ii. $[(x-1)]$
   iii. $[(x^2+1)]$
   iv. $[(x^2+1, 3)]$

   (b) Find a ring S and a function $f \in S$ such that f is zero at every point of $\operatorname{Spec} S$, but f is not the zero function. (HINT: *See HW9#1 from last semester.*)
3. Find rings R and S such that $R \not\cong S$, but

   (a) $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are homeomorphic (as topological spaces).

   (b) $|\operatorname{Spec} R|$ is in bijection with $|\operatorname{Spec} S|$, but $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are not homeomorphic.

   (HINT: *For (a), you can arrange so that $|\operatorname{Spec} R|$ and $|\operatorname{Spec} S|$ each consist of a single point; for (b), there's an example in which each set has two points.*)
4. Suppose $[\mathfrak{p}] \in \operatorname{Spec} R$. Show that $[\mathfrak{p}]$ is closed if and only if $\mathfrak{p}$ is a maximal ideal in R.
5. Suppose X is an affine variety over the algebraically closed field k, with coordinate ring $k[X]$. Show that there is a bijection between:
   - Closed points of the affine scheme $\operatorname{Spec} k[X]$.
   - Points of the variety X.

Go read the blog entry *Mumford's treasure map*, available at

`http://www.neverendingbooks.org/index.php/mumfords-treasure-map.html`