Homework 9

Due: Friday, April 6

1. Let A be a ring. For each $r, n \in \mathbb{Z}$, calculate $H^r(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n))$. (HINT: *Use the standard cover of* $\mathbb{P}^1$ *as the union of two affine lines.*)

2. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be homogeneous of degree d, and consider the curve $X = \mathcal{Z}_+(X) \subset \mathbb{P}^2_k$. Suppose $[1, 0, 0] \notin X$.

   Let $U_1 = X \cap \{X_1 \neq 0\}$ and let $U_2 = X \cap \{X_2 \neq 0\}$.

   (a) Show that $\mathcal{U} = \{U_1, U_2\}$ is an open cover of X.

   (b) Use the Cech complex $C^\bullet(\mathcal{U}, \mathcal{O}_X)$ to calculate the cohomology groups $H^\bullet(X, \mathcal{O}_X)$ explicitly. (HINT: *You should find that*

   $$\dim_k H^0(X, \mathcal{O}_X) = 1$$
   $$\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}.$$

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3. For a scheme X, let $\mathcal{O}_X^\times$ be the sheaf of abelian groups $U \mapsto \mathcal{O}_X(U)^\times$, which assigns to an open set U the group of invertible functions on U.

   Suppose X is separated and quasicompact.

   (a) Let $\mathcal{L}$ be a quasicoherent sheaf on X. In class, we said that $\mathcal{L}$ is invertible if and only if: There exists a cover $\mathcal{U} = \cup U_i$ of X, and elements $g_{ij} \in \mathcal{O}_X(U_{ij})^\times$, such that $g_{jk} \cdot g_{ij} = g_{ik} \in \mathcal{O}_X(U_{ijk})^\times$. Make sure you understand this.

   (b) For $\mathcal{L}$ and $\mathcal{U}$ as above, explain how to construct an element $\phi_{\mathcal{U}}(\mathcal{L})$ of $H^1_{\mathcal{U}}(X, \mathcal{O}_X^\times)$, and thus an element $\phi(\mathcal{L}) \in H^1(X, \mathcal{O}_X^\times)$. Show that

   $$\text{invertible sheaves on } X \xrightarrow{\phi} H^1(X, \mathcal{O}_X^\times)$$

   $$\mathcal{L} \longmapsto \phi(\mathcal{L})$$

   is a group homomorphism. What is $\ker \phi$?

   (c) Show that ϕ induces an isomorphism

   $$\mathrm{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times)$$

   (Remember, any element of $H^1(X, \mathcal{O}_X^\times)$ can be represented by a Cech cocycle on some open cover of X.)

   (d) Read Liu exercise 5.2.7, expecially 5.2.7.d.

4. (a) Liu 5.2.17.a. Show that the d defined here satisfies $d \circ d = 0$. Note: It suffices to assume that $\mathcal{U} = \mathcal{V}$; why?

   (b) Liu 5.2.17.b.