Homework 8
Due: Friday, March 30

1. Please read (at least the statements of) Liu, Chapter 5.1.4. See if the following interpretations make sense:

   (a) 5.1.24: Let $\mathcal{L}$ be a line bundle on X; let $s \in \mathcal{L}(U)$ be a section; and let $P \in U$. Even though $\mathcal{O}_{X,P} \cong \mathcal{L}_P$, the isomorphism is noncanonical, and so it doesn't make sense to evaluate s at P. Still, the vanishing or nonvanshing of s at P *does* make sense.

   In the special case where $\mathcal{L} = \mathcal{O}_X$, X_s is the complement of the vanishing locus of s.

   (b) 5.1.25: After twisting with a sufficiently high power of $\mathcal{L}$, a section which vanishes on some X_s is actually zero; and a section over some open set lifts.

   (c) 5.1.26: A very ample sheaf on X depends on a choice of embedding of X into some projective space. It's the pullback of Serre's sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$.

   (d) 5.1.31: If X is a scheme, and $\mathcal{L}$ is an invertible sheaf generated by $d+1$ global sections, then $\mathcal{L}$ determines a morphism into projective space.

   Concretely, let $s_0, \cdots, s_d \in \mathcal{L}(X)$. Morally, one can define a morphism

$$X \longrightarrow \mathbb{P}^d$$

$$P \longmapsto [s_0(P), \cdots, s_d(P)]$$

   (As noted before, something like $s_0(P)$ doesn't really make sense. But, *choose* an isomorphism $\mathcal{L}_P \to \mathcal{O}_{X,P}$. Then $s_i(P)/s_j(P)$ is well-defined, independently of this choice of isomorphism. See Liu for the actual definition of this morphism, which he defines locally on the open sets X_{s_i}.)

   (e) 5.1.34 An ample sheaf is a line bundle $\mathcal{L}$ such that some tensor power $\mathcal{L}^{\otimes m}$ defines is very ample.

   (f) 5.1.36 A scheme is quasiprojective if and only if it admits an ample sheaf.

2. Liu 5.1.12.cd.

3. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. Consider the following function on X:

$$\rho(P) = \dim_{\kappa(P)} \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P).$$

   (a) Show that ρ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{P \in X : \rho(P) \geq n\}$ is closed. (HINT: *Nakayama!*)

   (b) If $\mathcal{F}$ is locally free, and if X is connected, show that ρ is a constant function.

4. (a) Let A be a ring, and let $I \subset A$ be a principal. Think of I as an A-module, and consider the sheaf of modules $\widetilde{I}$ on Spec A. Show that $\widetilde{I}$ is an invertible sheaf.

(b) Let $A = \mathbb{Z}[\sqrt{-5}]$, and let $I = (2, 1 + \sqrt{-5})$. (Note that I is not principal!) Show that $\widetilde{I}$ is an invertible sheaf on $\operatorname{Spec} A$.

(HINT: *It suffices to check this on stalks. Suppose* $\mathfrak{p} \in \operatorname{Spec} A$. *On one hand, show that if* $I \not\subset \mathfrak{p}$, *then* $I_\mathfrak{p} = R_\mathfrak{p}$. *On the other hand, suppose* $I \subset \mathfrak{p}$. *Show that* $2 \in I^2$, *and thus* $2 \in I\mathfrak{p}$. *Now use Nakayama's lemma to show that* $I_\mathfrak{p} = (1 + \sqrt{-5})\mathfrak{p}$.)

More generally, let A be a Dedekind ring (such as the ring of integers in a number field), with field of fractions K. A fractional ideal is a sub-A-module $M \subset K$ *such that there exists some* $d \in A$ *with* $dM \subseteq A$. *Every fractional ideal determines an invertible sheaf on* $\operatorname{Spec} A$. *See, e.g., [Atiyah-Macdonald, Chapter 9]*