Homework 7

Due: Friday, March 9

1. Liu, 4.2.7. (HINT: *See last semester, HW12#1*)
2. Liu, 4.2.3. *Ignore normality.*
3. Let A be a B-algebra; we identify B with its image in A. Consider the homomorphism

$$\begin{array}{ccc} A \otimes_B A & \xrightarrow{f} & A \\ a_1 \otimes a_2 & \longmapsto & a_1 a_2 \end{array}$$

(This corresponds to $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A$.) Its kernel $I = \ker(f)$ is generated by elements of the form $a \otimes 1 - 1 \otimes a$.

Think of $A \otimes_B A$ as an A-module via left multiplication. Consider the map of A-modules

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{d}} & A \otimes_B A \\ a & \longmapsto & 1 \otimes a - a \otimes 1. \end{array}$$

   (a) Show that $\widetilde{d}(A) \subseteq I \subseteq A \otimes_B A$.
   (b) Show that $\widetilde{d}$ is additive; $\widetilde{d}(a_1 + a_2) = \widetilde{d}(a_1) + \widetilde{d}(a_2)$.
   (c) Show that $\widetilde{d}(B) = 0$.
   (d) Show that $\widetilde{d}(a_1 a_2) - (a_1 \widetilde{d}(a_2) + a_2 \widetilde{d}(a_1)) \in I^2$.
   (e) Conclude that the composition d :

$$A \longrightarrow A \otimes_B A \longrightarrow I/I^2$$

   is an element of $\operatorname{Der}_B(A, I/I^2)$, the set of derivations from A to I/I^2 which vanish on B. (See 10.1.2 from last semester for the definition of a derivation in the special case $B = k$.)

Remark: In fact, this is a universal *derivation, in the sense that if M is any A-module, and if $d' : A \to M$ is a B-linear derivation, then d' factors uniquely:*

$$\begin{array}{ccc} A & \xrightarrow{d'} & M \\ & \searrow^{d} & \uparrow \\ & & I/I^2 \end{array}$$

Professor Jeff Achter
Colorado State University