Homework 5

Due: Friday, February 23

1. Please read:
   (a) Last semester, HW7#4.
   (b) Last semester, 2.5 and 6.2.
   (c) Liu, 2.5, statements and examples.
2. Let X be an integral scheme, and let $\eta \in X$ be a generic point.
   (a) Show that the local ring $\mathcal{O}_{X,\eta}$ is a field.
   (b) Suppose $U = \operatorname{Spec} R$ is an open affine subset of X. Show that $\mathcal{O}_{X,\eta} \cong \operatorname{Frac} R$.
3. Liu 2.4.11. (HINT: *For the equivalence of (i), (ii) and (iii), use HW2#1b and the remark after 2.2.12, which we proved in class. For the rest, you may want to use the fact (not hard to prove) that if $f : X \to Y$ is a continuous map, and if $U \subset V \subset X$ with U dense in V, then $f(U)$ is dense in $f(V)$.*)
4. Let X, Y and T be S-schemes. Use the universal property of fiber products to show that there is a natural bijection of sets

$$(X \times_S Y)(T) \longrightarrow X(T) \times_{S(T)} Y(T)$$

   (HINT: *Compare Liu, Example 3.1.6, page 81.*)
5. (a) Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.
   (b) Let $X = \operatorname{Spec} k[x]$, $Y = \operatorname{Spec} k[y]$, and let ϕ be the morphism $X \to Y$ attached to

$$k[y] \longrightarrow k[x]$$

$$y \longrightarrow x^2$$

   Show that $X \times_Y X$ has two irreducible components.