1. (a) Liu 2.4.4
(b) Find an example of a scheme which is connected but not irreducible.

2. Let $S$ and $T$ be rings, and consider the ring $S \oplus T$. Note that the natural surjection $S \oplus T \to S$ corresponds to a closed immersion $\text{Spec } S \to \text{Spec}(S \oplus T)$.

Show that, as subscheme of $\text{Spec}(S \oplus T)$, $\text{Spec } S$ is both open and closed. (HINT: Consider $D((1, 0))$.)

3. Let $R$ be a ring. An element $e \in R$ is called an idempotent if $e \neq 1$ but $e^2 = e$.

For example, if $S$ and $T$ are rings, then the element $(1, 0) \in S \oplus T$ is an idempotent of $S \oplus T$.

(a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2 - x)$.
(b) Let $e$ be an idempotent of $R$. Show that the map

$$R \longrightarrow eR \oplus (1 - e)R$$

$$a \longmapsto (ea, (1 - e)a)$$

is an isomorphism.
(c) Show that $\text{Spec } R$ is not connected if and only if $R$ has an idempotent.

4. Suppose $S$ and $T$ and $U$ are sets and that there are maps of sets $f : S \to U$ and $g : T \to U$.

The fiber product of $S$ and $T$ over $U$, denoted $S \times_{f, U, g} T$ or $S \times_U T$ if the maps are understood, is

$$S \times_U T := \{(s, t) : s \in S, t \in T, f(s) = g(t)\}.$$ 

This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it’s the smallest set which makes the following diagram commute:

$$\begin{array}{ccc}
S \times_U T & \longrightarrow & T \\
\downarrow^{g} & & \downarrow^{g} \\
S & \longrightarrow & U \\
\downarrow^{f} & & \\
S & \longrightarrow & U
\end{array}$$

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M673: Algebraic geometry
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(a) Consider the following sets:

\[ S = \{1, 2, 3, 4\} \]
\[ T = \{a, b, c, d, e, f\} \]
\[ U = \{\alpha, \beta\} \]
\[ V = \{\gamma\} \]

and the following maps between them:

\[ S \xrightarrow{f} U \quad T \xrightarrow{g} U \]

\[ \begin{array}{c}
1, 2 \quad \xrightarrow{\alpha} \\
\downarrow \quad \downarrow \\
3, 4 \quad \xrightarrow{\beta}
\end{array} \quad \begin{array}{c}
a, b, c \quad \xrightarrow{\alpha} \\
\downarrow \quad \downarrow \\
d, e, f, g \quad \xrightarrow{\beta}
\end{array} \]

\[ S \xrightarrow{p} V \quad T \xrightarrow{q} V \]

anything \quad \xrightarrow{\gamma} \quad anything \quad \xrightarrow{\gamma}

What is \( S \times_U T \)? What is \( S \times_V T \)?

(b) Suppose \( S \) and \( T \) are arbitrary sets, \( U \) is a set with one element, and \( f : S \to U \) and \( g : T \to U \) are the maps which send everything to the (unique) element of \( U \). What is \( S \times_U T \)?

(c) Suppose \( S \) is a subset of \( T \), and consider the inclusion \( \iota : S \hookrightarrow T \) and the identity map \( \text{id} : T \to T \). What is \( S \times_{U,T,\text{id}} T \)?