Homework 4

Due: Friday, February 16

1. (a) Liu 2.4.4
   (b) Find an example of a scheme which is connected but not irreducible.

2. Let S and T be rings, and consider the ring $S \oplus T$. Note that the natural surjection $S \oplus T \to S$ corresponds to a closed immersion $\operatorname{Spec} S \to \operatorname{Spec}(S \oplus T)$.

   Show that, as subscheme of $\operatorname{Spec}(S \oplus T)$, $\operatorname{Spec} S$ is both open and closed. (HINT: *Consider* $D((1,0))$.)

3. Let R be a ring. An element $e \in R$ is called an *idempotent* if $e \neq 1$ but $e^2 = e$.

   For example, if S and T are rings, then the element $(1,0) \in S \oplus T$ is an idempotent of $S \oplus T$.

   (a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2 - x)$.
   (b) Let e be an idempotent of R. Show that the map

   $$R \longrightarrow eR \oplus (1-e)R$$

   $$a \longmapsto (ea, (1-e)a)$$

   is an isomorphism.
   (c) Show that $\operatorname{Spec} R$ is not connected if and only if R has an idempotent.

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4. Suppose S and T and U are *sets* and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of S and T over U, denoted $S \times_{f,U,g} T$ or $S \times_U T$ f the maps are understood, is

   $$S \times_U T := \{(s,t) : s \in S, t \in T, f(s) = g(t)\}.$$

   This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it's the smallest set which makes the following diagram commute:

   $$\begin{array}{ccc} S \times_U T & \longrightarrow & T \\ \downarrow & & \downarrow g \\ S & \xrightarrow{f} & U \end{array}$$

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(a) Consider the following sets:

$$\begin{aligned} S &= \{1, 2, 3, 4\} \\ T &= \{a, b, c, d, e, f\} \\ U &= \{\alpha, \beta\} \\ V &= \{\gamma\} \end{aligned}$$

and the following maps between them:

$$\begin{array}{rcl@{\qquad}rcl} S & \xrightarrow{f} & U & T & \xrightarrow{g} & U \\ 1, 2 & \longmapsto & \alpha & a, b, c & \longmapsto & \alpha \\ 3, 4 & \longmapsto & \beta & d, e, fg & \longmapsto & \beta \end{array}$$

$$\begin{array}{rcl@{\qquad}rcl} S & \xrightarrow{p} & V & T & \xrightarrow{q} & V \\ \text{anything} & \longmapsto & \gamma & \text{anything} & \longmapsto & \gamma \end{array}$$

What is $S \times_U T$? What is $S \times_V T$?

(b) Suppose S and T are arbitrary sets, U is a set with one element, and $f : S \to U$ and $g : T \to U$ are the maps which send everything to the (unique) element of U. What is $S \times_U T$?

(c) Suppose S is a subset of T, and consider the inclusion $\iota : S \hookrightarrow T$ and the identity map $\mathrm{id} : T \to T$. What is $S \times_{\iota, T, \mathrm{id}} T$?