Homework 4 Due: Friday, February 16

- 1. (a) Liu 2.4.4
 - (b) Find an example of a scheme which is connected but not irreducible.
- Let *S* and *T* be rings, and consider the ring *S* ⊕ *T*. Note that the natural surjection *S* ⊕ *T* → *S* corresponds to a closed immersion Spec *S* → Spec(*S* ⊕ *T*).
 Show that, as subscheme of Spec(*S* ⊕ *T*), Spec *S* is both open and closed. (HINT: *Consider*

Show that, as subscheme of Spec($S \oplus I$), Spec S is both open and closed. (HINT: Consider D((1,0)).)

- 3. Let *R* be a ring. An element $e \in R$ is called an *idempotent* if $e \neq 1$ but $e^2 = e$. For example, if *S* and *T* are rings, then the element $(1,0) \in S \oplus T$ is an idempotent of $S \oplus T$.
 - (a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2 x)$.
 - (b) Let *e* be an idempotent of *R*. Show that the map

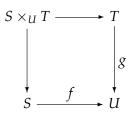
$$R \longrightarrow eR \oplus (1-e)R$$
$$a \longmapsto (ea, (1-e)a)$$

is an isomorphism.

- (c) Show that Spec *R* is not connected if and only if *R* has an idempotent.
- 4. Suppose *S* and *T* and *U* are *sets* and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of *S* and *T* over *U*, denoted $S \times_{f,U,g} T$ or $S \times_U T$ f the maps are understood, is

$$S \times_U T := \{(s,t) : s \in S, t \in T, f(s) = g(t)\}.$$

This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it's the smallest set which makes the following diagram commute:



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M673: Algebraic geometry Spring 2007 (a) Consider the following sets:

$$S = \{1, 2, 3, 4\}$$

$$T = \{a, b, c, d, e, f\}$$

$$U = \{\alpha, \beta\}$$

$$V = \{\gamma\}$$

and the following maps between them:

$S \xrightarrow{f} U$	$T \xrightarrow{g} U$
$1,2 \longmapsto \alpha$	$a, b, c \longmapsto \alpha$
$3,4 \longmapsto \beta$	$d, e, fg \longmapsto \beta$
$S \xrightarrow{p} V$	$T \xrightarrow{q} V$
anything $\longmapsto \gamma$	anything $\longmapsto \gamma$

What is $S \times_U T$? What is $S \times_V T$?

- (b) Suppose *S* and *T* are arbitrary sets, *U* is a set with one element, and $f : S \rightarrow U$ and $g : T \rightarrow U$ are the maps which send everything to the (unique) element of *U*. What is $S \times_U T$?
- (c) Suppose *S* is a subset of *T*, and consider the inclusion $\iota : S \hookrightarrow T$ and the identity map id : $T \to T$. What is $S \times_{\iota,T,id} T$?

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