Homework 2

Due: Friday, February 2

1. *Liu 2.3.3*

   (a) Let U be a subset of an affine scheme $\operatorname{Spec} R$. Show that the closure $\overline{U}$ of U in $\operatorname{Spec} R$ is $\mathcal{Z}(I)$, where $I = \cap_{\mathfrak{p} \in U} \mathfrak{p}$.

   (b) Let $\phi : R \to S$ be a ring homomorphism. Let $f = \operatorname{Spec}(\phi) : \operatorname{Spec} S \to \operatorname{Spec} R$. Show that the closure of $\operatorname{im}(f)$ is $\mathcal{Z}(\ker \phi)$.

2. Describe the points and the sheaf of functions of each of the following schemes.

   (a) $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$

   (b) $X_2 = \operatorname{Spec} \mathbb{C}[x]/(x^2 - x)$

   (c) $X_3 = \operatorname{Spec} \mathbb{C}[x]/(x^3 - x^2)$

   (d) $X_4 = \operatorname{Spec} \mathbb{R}[x]/(x^2 + 1)$

3. *Constant sheaves*

   (a) Let X be a topological space; let A be a finite abelian group with more than one element; and consider the presheaf $\mathcal{F}$ on X given by $\mathcal{F}(U) = A$. Show that $\mathcal{F}$ is a sheaf if and only if each open subset of X is connected.

   (b) There is a sheaf $\mathcal{G}$ such that for every *connected* open subset $U \subseteq X$, $\mathcal{G}(U) = A$. How would you "fix" this to make a sheaf $\mathcal{G}$ such that for every connected open $U \subseteq X$, $\mathcal{G}(U) = A$?

4. *Support* Let $\mathcal{F}$ be a sheaf on X, let $U \subset X$ be an open subset, and let $s \in \mathcal{F}(U)$ be a section.

   (a) Suppose $P \in U$. Show that the stalk of s at P, s_P, is zero if and only if there is some open neighborhood $P \in V \subseteq U$ such that $\operatorname{res}_{V,U}(s) = 0 \in \mathcal{F}(V)$.

   (b) The support of s is

   $$\operatorname{supp}(s) = \{P \in U : s_P \neq 0\}.$$

   Show that $\operatorname{supp}(s)$ is a closed subset of U.

5. *Skyscraper sheaves* Let X be a topological space, and let A be a finite abelian group. Let $P \in X$ be a point whose closure is Z, and define a sheaf $\mathcal{S} = \mathcal{S}_P(A)$ as follows:

   $$\mathcal{S}(U) = \begin{cases} A & P \in U \\ \{0\} & P \notin U \end{cases}$$

   (a) Compute the stalks of $\mathcal{S}$:

   $$\mathcal{S}_Q = \begin{cases} A & Q \in Z \\ \{0\} & \text{otherwise} \end{cases}.$$

(b) Let $i : Z \to X$ be the inclusion of topological spaces. Note that Z is connected. Let $\mathcal{F}$ be the constant sheaf associated to A. Show that

$$i_*(\mathcal{F}) = \mathcal{S}.$$