## Homework 12

Due: Friday, April 27

1. Let $X/k$ be a smooth projective curve, and let $P \in X$ be a point (of codimension one). Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at $P$.

2. Let $f : X \to Y$ be a finite morphism of smooth, projective curves. Show that $g_X \geq g_Y$. (HINT: *You may assume $f$ is separable.*)

3. Let $X/\mathbb{F}_q$ be a smooth projective curve of genus $g$. Suppose that $n \geq 2g - 1$, and let $D \in \mathrm{Div}(X)$ be a divisor of degree $n$. Show that the number of effective divisors linearly equivalent to $D$ is
$$\frac{q^{n-g+1} - 1}{q - 1}.$$
(HINT: *How many elements are in $\mathbb{P}^m(\mathbb{F}_q)$?*)

4. Let $X/\mathbb{F}_q$ be a smooth projective curve. We will define the zeta function of $X$ in the following way:
$$\begin{aligned} a_X(n) &= \#\{D \in \mathrm{Div}(X) : D \geq 0, \deg(D) = n\} \\ Z_X(t) &= \sum_{n \geq 0} a_X(n) t^n \end{aligned}$$
If $Y/\mathbb{F}_q$ is an arbitrary variety, for each $r \geq 1$ let $N_Y(r) = \#Y(\mathbb{F}_{q^r})$. One defines the zeta function of $Y$ as
$$\widetilde{Z}_Y(t) = \exp\Big(\sum_{r \geq 1} N_Y(r) \frac{t^r}{r}\Big).$$
Show that these two definitions are the same, i.e., that
$$Z_X(t) = \widetilde{Z}_X(t).$$
(HINT: *Don't worry about convergence of any of the power series in question.*)

5. Using which ever definition you prefer, show that
$$Z_{\mathbb{P}^1/\mathbb{F}_q}(t) = \frac{1}{(1-t)(1-qt)}.$$