Homework 10

Due: Friday, April 13

1. Let X be a projective variety over $\operatorname{Spec} k$. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, its Euler characteristic is
$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$
Suppose
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
is an exact sequence of coherent sheaves. Show that
$$\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H}).$$
(HINT: *See Liu, 5.3.26 and 7.3.16. We won't use this for a while.*)

2. As in HW9#2, let $F \in k[X_0, X_1, X_2]$ be homogeneous of degree d, and let $X = \mathcal{Z}_+(F) \subset \mathbb{P}^2$.
   (a) Suppose $n \gg d$. Calculate $\dim_k H^0(X, \mathcal{O}_X(n))$ and $\dim_k H^1(X, \mathcal{O}_X(n))$.
   (b) Find a numerical polynomial $P_X(T)$ such that for $n \gg 0$, $P_X(n) = \chi(\mathcal{O}_X(n))$.
   (c) What is $1 - P_X(0)$? Read [672 HW 13#4] and [Liu, Exercise 7.3.4]

3. Let K be a field, and let L be a finite simple extension; then $L \cong K[Y]/(f(Y))$ for some polynomial f.
Show that
$$\Omega_{L/K} \cong \begin{cases} 0 & L/K \text{ separable} \\ L & L/K \text{ inseparable} \end{cases}.$$

4. Let B be an A-algebra. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal, with residue field $\kappa(\mathfrak{p}) = A/\mathfrak{p}$. What is $\Omega_{B \otimes \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$? (HINT: *See Liu 6.1.8.a.*)

5. Let $B = \mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[Y]/(Y^2 + 5)$, and consider $\pi : \operatorname{Spec}(B) \to \operatorname{Spec} \mathbb{Z}$. Describe the $\mathbb{Z}$-module $\pi_*(\Omega_{B/\mathbb{Z}})$.