Homework 1

Due: Friday, January 26

1. Find rings R and S such that $R \not\cong S$, but
   (a) $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are homeomorphic (as topological spaces).
   (b) $|\operatorname{Spec} R|$ is in bijection with $|\operatorname{Spec} S|$, but $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are not homeomorphic.

   (HINT: *For (a), you can arrange so that $|\operatorname{Spec} R|$ and $|\operatorname{Spec} S|$ each consist of a single point; for (b), there's an example in which each set has two points.*)

2. Suppose $[\mathfrak{p}] \in \operatorname{Spec} R$. Show that $[\mathfrak{p}]$ is closed if and only if $\mathfrak{p}$ is a maximal ideal in R.

3. Suppose X is an affine variety over the algebraically closed field k, with coordinate ring $k[X]$. Show that there is a bijection between:
   - Closed points of the affine scheme $\operatorname{Spec} k[X]$.
   - Points of the variety X.

4. Let $\phi : R \to S$ be a ring homomorphism.
   (a) Suppose R and S are finitely generated k-algebras, where k is an algebraically closed field. Show that if $Q \in \operatorname{Spec} S$ is a closed point, then $\operatorname{Spec}(\phi)(Q)$ is a closed point of $\operatorname{Spec} R$.
   (b) Give an example where $Q \in \operatorname{Spec} S$ is closed, but $\operatorname{Spec}(\phi)(Q)$ is *not* closed.

   (HINT: *See HW2#2 from last semester.*)

5. Let $\phi : X \to Y$ be a continuous map of topological spaces. Suppose $\mathcal{F}$ is a sheaf on X. Define $\phi_*\mathcal{F}$, the pushforward of $\mathcal{F}$ by ϕ, as follows: For each open $V \subset Y$, set

$$(\phi_*\mathcal{F})(V) = \mathcal{F}(\phi^{-1}(V)).$$

   Prove that $\phi_*\mathcal{F}$ is a sheaf on Y.