Homework 9
Due: Friday, April 7

1. Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes, and let $\mathcal{F}$ be a quasicoherent sheaf on $\operatorname{Spec} B$. Show that $f_*\mathcal{F}$ is a quasicoherent sheaf (on $\operatorname{Spec} A$).

2. Let X be a locally Noetherian scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

   For $P \in X$, the stalk $\mathcal{F}_P$ is a module over $\mathcal{O}_{X,P}$. Define

   $$\mathcal{F}(P) = \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P),$$

   a vector space over the residue field $\kappa(P)$, and let

   $$\operatorname{rank}_P \mathcal{F} = \dim_{\kappa(P)} \mathcal{F}(P).$$

   (a) Suppose $s_1, \cdots, s_n \in \mathcal{F}_P$ span $\mathcal{F}(P)$. Show that $s_1, \cdots, s_n$ generate $\mathcal{F}_P$ over $\mathcal{O}_{X,P}$. (HINT: *Nakayama.*)

   (b) Show there is some open neighborhood U of P such that there is a surjective homomorphism

   $$(\mathcal{O}_X|_U)^n \xrightarrow{(s_1, \cdots, s_n)} \mathcal{F}|_U$$

   of sheaves on U.

   (c) Show that the function $P \mapsto \operatorname{rank}_P \mathcal{F}$ is upper semicontinuous. (HINT: *Equivalently, show that for each n,*

   $$\{P \in X : \operatorname{rank}_P \mathcal{F} \le n\}$$

   is open.)

3. [GW] 7.7.

4. [GW] 7.10.