Homework 8

Due: Friday, March 31

1. [GW]6.16. *Identify those points where the morphism is not smooth.*

2. (a) [GW]6.12(a).

   (b) Give an example of a field k, a scheme $X \to \operatorname{Spec} k$ locally of finite type such that X is regular, but not geometrically regular.

   Compare with [GW] Corollary 6.32.

3. Use the valuative criterion to show that $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper.

A morphism $f : Y \to X$ of schemes is *finite* if for every open affine subset $U \subseteq X$, $f^{-1}(U)$ is affine and $\mathcal{O}_Y(f^{-1}(U))$ is a finite module over $\mathcal{O}_X(U)$.

(Equivalently, there exists some open affine cover $X = \cup U_\alpha = \cup \operatorname{Spec} A_\alpha$ such that $f^{-1}(U_\alpha) = \operatorname{Spec} B_\alpha$ is affine and B_α is a finite module over A_α.)

A morphism is *quasifinite* if, for each $P \in X$, $f^{-1}(P)$ is a finite set.

4. (a) Show that any finite morphism is quasifinite.

   (b) Show that the converse is false.(HINT: *672HW3#3b*).

5. (a) Show that the property of being quasifinite is stable under base change.

   (b) Show that the property of being finite is stable under base change.

6. Consider a finite morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$. A version of the going-up theorem says:

   Theorem Suppose that $A \subset B$ is an integral extension of rings. Suppose $\mathfrak{p} \subset A$ is prime; then there exists a prime $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$. Moreover, $\mathfrak{q}$ may be chosen to contain any given ideal I which satisfies the (obvious necessary) condition $I \cap A \subseteq \mathfrak{p}$.

   (This formulation comes from Eisenbud's *Commutative algebra*, Prop. 4.15; see also 672HW12#1.)

   (a) Suppose that f induces an inclusion $A \subset B$; show that f is a closed map.

   (b) Continue to suppose $B \subset A$; show that f is proper. (HINT: *5.(b)*)

Problems 5 and 6 can be used to assemble a proof of:

Theorem A finite morphism is proper.