Homework 6

Due: Friday, March 3

1. [GW] 4.15.

2. Let $X = \operatorname{Spec} \mathbb{Z}[x]/(x^2+1)$ and $Y = \operatorname{Spec} \mathbb{Z}$; consider the map $X \to Y$ which comes from $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]/(x^2+1)$.

   Suppose $Q \in Y$. Describe the fiber X_Q. (HINT: *Your answer will depend on* $Q = [(q)]$.)

3. (a) Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.

   (b) Let $X = \operatorname{Spec} k[x]$, $Y = \operatorname{Spec} k[y]$, and let ϕ be the morphism $X \to Y$ attached to

$$k[y] \longrightarrow k[x]$$

$$y \longmapsto x^2$$

   i. Suppose $\operatorname{char}(k) \neq 2$. Show that $X \times_Y X$ has two irreducible components.

   ii. Suppose $\operatorname{char}(k) = 2$. Describe $X \times_Y X$.

   (HINT: *It may be marginally easier to think of two copies of* X, X_1 *and* X_2, *with coordinates* x_1 *and* x_2...)

4. *Weil restriction* Let $T \to S$ and $X \to T$ be schemes. The Weil restriction $\mathbf{R}_{T/S}(X)$ is the scheme such that, for each S-scheme $Z \to S$,

$$\mathbf{R}_{T/S}(X)(Z) = \operatorname{Mor}_{(}\, T \times_S Z, X).$$

   (If $T = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$, this is usually just written as $\mathbf{R}_{B/A}(X)$.)

   Let $\mathbb{G}_m = \operatorname{Spec} \mathbb{R}[s,t]/(st-1) \cong \operatorname{Spec} \mathbb{R}[s, s^{-1}]$.

   (a) Show that $\mathbb{G}_m$ represents the functor on $\mathbb{R}$-schemes $Z \to \mathcal{O}_Z(Z)^\times$ (the units in $\mathcal{O}_Z(Z)$).

   (b) Show that

$$\mathbf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \cong \operatorname{Spec}(\mathbb{R}[x, y, \frac{1}{x^2+y^2}]).$$

   (Of course, $\mathbb{R}[x, y, (x^2+y^2)^{-1}]$ means $\mathbb{R}[x,y,z]/(z(x^2+y^2)-1)$.)