Homework 5

Due: Friday, February 24

Even though the homework isn't due until February 24, please do problem 1 before class on Wednesday, February 22.

1. Suppose S and T and U are *sets* and that there are maps of sets $f : S \to U$ and $g : T \to U$. The fiber product of S and T over U, denoted $S \times_{f,U,g} T$ or $S \times_U T$ if the maps are understood, is

$$S \times_U T := \{(s,t) : s \in S, t \in T, f(s) = g(t)\}.$$

This set comes with projections $S \times_U T \to S$ and $S \times_U T \to T$, and in some sense it's the smallest set which makes the following diagram commute:

$$\begin{array}{ccc} S \times_U T & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\;f\;} & U \end{array}$$

(a) Consider the following sets:

$$\begin{aligned} S &= \{1,2,3,4\} \\ T &= \{a,b,c,d,e,f\} \\ U &= \{\alpha,\beta\} \\ V &= \{\gamma\} \end{aligned}$$

and the following maps between them:

$$\begin{array}{cccccc} S & \xrightarrow{\;f\;} & U & T & \xrightarrow{\;g\;} & U \\ 1,2 & \longmapsto & \alpha & a,b,c & \longmapsto & \alpha \\ 3,4 & \longmapsto & \beta & d,e,fg & \longmapsto & \beta \end{array}$$

$$\begin{array}{cccccc} S & \xrightarrow{\;p\;} & V & T & \xrightarrow{\;q\;} & V \\ \text{anything} & \longmapsto & \gamma & \text{anything} & \longmapsto & \gamma \end{array}$$

What is $S \times_U T$? What is $S \times_V T$?

(b) Suppose S and T are arbitrary sets, U is a set with one element, and $f : S \to U$ and $g : T \to U$ are the maps which send everything to the (unique) element of U. What is $S \times_U T$?

(c) Suppose S and T are subsets of U; denote the inclusions by $i : S \hookrightarrow U$ and $j : T \hookrightarrow U$. What is $S \times_{i,U,j} T$?

2. Consider the following functor:

$$\mathsf{Sch}^{\mathrm{opp}} \xrightarrow{\quad F \quad} \mathsf{Set}$$

$$Z \longmapsto \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{O}_Z(Z) \text{ and } ad - bc \in \mathcal{O}_Z(Z)^\times \right\}$$

Let X be the scheme

$$X = \operatorname{Spec} \frac{\mathbb{Z}[a, b, c, d, z]}{z \cdot (ad - bc) = 1}.$$

Show that X represents the functor F, i.e., for every Z we have

$$X(Z) = F(Z).$$

3. [GW] 3.23.

4. [GW] 4.1. *You need only give one proof.* (HINT: *Elements of A are in bijection with ring homomorphisms $R[T]- > A$.*)