Homework 4

Due: Friday, February 17

1. [GW]3.5.

2. [GW]3.6.

3. [GW]3.7.

4. Let G be a finite group which acts on a ring A. (Note that G then acts on $\operatorname{Spec} A$, as well.) Let A^G be the ring of G-invariants:

$$A^G = \{a \in A : \forall \sigma \in G, a^\sigma = a\}.$$

The inclusion $A^G \hookrightarrow A$ induces a morphism of affine schemes $\varpi : \operatorname{Spec} A \to \operatorname{Spec} A^G$.

   (a) Show that A is integral over A^G. (HINT: *If $a \in A$, consider $P_a(T) = \prod_{\sigma \in G}(T - a^\sigma)$.*)

   (b) Show that $\varpi : \operatorname{Spec} A \to \operatorname{Spec} A^G$ is surjective. (HINT: *Use the going-up theorem used on HW12 last semester.*)

   (c) Suppose $\varpi([\mathfrak{q}]) = \varpi([\mathfrak{r}])$. Show that $\mathfrak{q} = \mathfrak{r}^\sigma$ for some $\sigma \in G$. If you like, you may proceed as follows:

      i. Explain why $\mathfrak{q} \cap A^G = \mathfrak{r} \cap A^G =: \mathfrak{p}$.

      ii. Choose some $a \in \mathfrak{q}$. Show that $\prod_{\sigma \in G} a^\sigma \in \mathfrak{p}$.

      iii. Show there is some $\tau \in G$ such that $a^\tau \in \mathfrak{r}$.

      iv. Use the prime avoidance lemma to show that $\mathfrak{r} = \mathfrak{q}^\sigma$ for some $\sigma \in G$.

   (d) Conversely, suppose $\mathfrak{q} = \mathfrak{r}^\sigma$ for some σ. Show that $\varpi([\mathfrak{q}]) = \varpi([\mathfrak{r}])$.

5. *These parts are essentially unrelated, but neither is very difficult.*

   (a) Suppose that W is a subscheme of X. Explain why, for any field K, there is an inclusion of sets $W(K) \hookrightarrow X(K)$.

   (b) Consider $X = \operatorname{Spec}(\mathbb{F}_{p^r})$ (as a scheme over $\operatorname{Spec} \mathbb{F}_p$). What is $\#X(\mathbb{F}_{p^m})$? (HINT: *The answer depends on whether $r|m$.*)