Homework 3

Due: Friday, February 10

1. Let X be the affine scheme
$$X = \operatorname{Spec} \frac{\mathbb{Z}[x,y,z]}{(x^2+y^2-z^2)}.$$
Let B be any ring. Carefully explain the bijection between:
   - $\operatorname{Mor}(\operatorname{Spec} B, X)$, and
   - triples $\alpha, \beta, \gamma \in B$ such that $\alpha^2 + \beta^2 = \gamma^2$.

2. Let A be an an integral domain, $X = \operatorname{Spec} A$, and $\eta = [(0)] \in X$. Use the definition of a stalk [GW (2.6)] to compute $\mathcal{O}_{X,[(0)]}$.

   See if you can relate this to last semester's field of rational functions on an irreducible variety.

3. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, and let $U = \mathbb{A}^2 - \{[(x,y)]\}$.
Let $V_1 = D(x)$ and $V_2 = D(y)$.
   (a) Verify that $U = V_1 \cup V_2$ is an open cover of U.
   (b) Calculate $\mathcal{O}_X(V_1)$, $\mathcal{O}_X(V_2)$ and $\mathcal{O}_X(V_1 \cap V_2)$.
   (c) Use this to calculate $\mathcal{O}_X(U)$.

   If U were affine, we would have $U = \operatorname{Spec}(\mathcal{O}_X(U))$; but this isn't true!

4. Let A and B be rings, and consider the ring $A \oplus B$. Note that the natural surjection $A \oplus B \to A$ corresponds to a closed immersion $\operatorname{Spec} A \hookrightarrow \operatorname{Spec}(A \oplus B)$.

   Show that, as subscheme of $\operatorname{Spec}(A \oplus B)$, $\operatorname{Spec} A$ is both open and closed. (HINT: *Consider* $D((1,0))$.)

5. Let A be a ring. An element $e \in A$ is called a nontrivial idempotent if $e^2 = e$ but $e \notin \{0,1\}$.

   For example, the element $(1,0) \in B \oplus C$ is a nontrivial idempotent of $B \oplus C$.

   (a) List all the idempotents of the ring $\mathbb{C}[x]/(x^2 - x)$.
   (b) Let e be an idempotent of A. Show that the map
$$A \longrightarrow eA \oplus (1-e)A$$
$$a \longmapsto (ea, (1-e)a)$$
is an isomorphism.
   (c) Show that $\operatorname{Spec} A$ is not connected if and only if A has a nontrivial idempotent. (HINT: *Suppose* $\operatorname{Spec} A = Z(I) \cup Z(J)$ *with* $Z(I) \cap Z(J) = \emptyset$; *show* $\operatorname{Spec}(A)$ *is homeomorphic to* $\operatorname{Spec}(A/I \oplus A/J)$.)

   Alternatively, do [GW 2.18].