Homework 10

Due: Friday, May 5

1. Let A be a ring. For each $n \in \mathbb{Z}$, calculate $H^1(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(n))$. (HINT: *Use the standard cover of* $\mathbb{P}^1$ *as the union of two affine lines.*)

2. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be homogeneous of degree d, and consider the curve $X = \mathcal{Z}_+(X) \subset \mathbb{P}^2_k$. Suppose $[1,0,0] \notin X$.

   Let $U_1 = X \cap \{X_1 \neq 0\}$ and let $U_2 = X \cap \{X_2 \neq 0\}$.

   (a) Show that $\mathcal{U} = \{U_1, U_2\}$ is an open cover of X.

   (b) Use the Cech complex $C^\bullet(\mathcal{U}, \mathcal{O}_X)$ to calculate the cohomology groups $H^\bullet(X, \mathcal{O}_X)$ explicitly. (HINT: *You should find that*

$$\dim_k H^0(X, \mathcal{O}_X) = 1$$
$$\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}.$$

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   One way of defining the genus of a smooth, projective curve X is $\dim H^1(X, \mathcal{O}_X)$.

3. For a scheme X, let $\mathcal{O}_X^\times$ be the sheaf of abelian groups $U \mapsto \mathcal{O}_X(U)^\times$, which assigns to an open set U the group of invertible functions on U.

   Suppose X is separated and quasicompact.

   (a) Let $\mathcal{L}$ be a quasicoherent sheaf on X. In class, we said that $\mathcal{L}$ is invertible if and only if: There exists a cover $\mathcal{U} = \cup U_i$ of X, and elements $g_{ij} \in \mathcal{O}_X(U_{ij})^\times$, such that $g_{jk} \cdot g_{ij} = g_{ik} \in \mathcal{O}_X(U_{ijk})^\times$. Make sure you understand this.

   (b) For $\mathcal{L}$ and $\mathcal{U}$ as above, explain how to construct an element $\phi_{\mathcal{U}}(\mathcal{L})$ of $H^1(\mathcal{U}, \mathcal{O}_X^\times)$, and thus an element $\phi(\mathcal{L}) \in H^1(X, \mathcal{O}_X^\times)$. Show that

$$\text{invertible sheaves on } X \xrightarrow{\phi} H^1(X, \mathcal{O}_X^\times)$$
$$\mathcal{L} \longmapsto \phi(\mathcal{L})$$

   is a group homomorphism. What is $\ker \phi$?

   (c) Show that ϕ induces an isomorphism

$$\operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times)$$

   (Remember, any element of $H^1(X, \mathcal{O}_X^\times)$ can be represented by a Cech cocycle on some open cover of X.)

4. *For students in Math 605*

   (a) Let A be a ring, and let $I \subset A$ be a principal. Think of I as an A-module, and consider the sheaf of modules $\widetilde{I}$ on Spec A. Show that $\widetilde{I}$ is an invertible sheaf.

   (b) Let $A = \mathbb{Z}[\sqrt{-5}]$, and let $I = (2, 1 + \sqrt{-5})$. (Note that I is not principal!) Show that $\widetilde{I}$ is an invertible sheaf on Spec A.

   (HINT: *It suffices to check this on stalks. Suppose* $\mathfrak{p} \in \operatorname{Spec} A$. *On one hand, show that if* $I \not\subset \mathfrak{p}$, *then* $I_{\mathfrak{p}} = R_{\mathfrak{p}}$. *On the other hand, suppose* $I \subset \mathfrak{p}$. *Show that* $2 \in I^2$, *and thus* $2 \in I\mathfrak{p}$. *Now use Nakayama's lemma to show that* $I_{\mathfrak{p}} = (1 + \sqrt{-5})\mathfrak{p}$.)

   More generally, let A be a Dedekind ring (such as the ring of integers in a number field), with field of fractions K. A fractional ideal is a sub-A-module $M \subset K$ *such that there exists some* $d \in A$ *with* $dM \subseteq A$. *Every fractional ideal determines an invertible sheaf on* Spec A. *See, e.g., [Atiyah-Macdonald, Chapter 9]*

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5. [Katz] Let $N = 2n + 1$, and consider $\mathbb{P}^N$ with coordinates $X_0, \cdots, X_n, Y_0, \cdots, Y_n$. Let

$$F = \sum_{i=0}^{n} X_i Y_i^q - X_i^q Y_i,$$

   and let $Z/\mathbb{F}_q$ be the (smooth, irreducible, projective) hypersurface

$$Z = \mathcal{Z}_{\mathbb{P}^N_{\mathbb{F}_q}}(F).$$

   Show that for every hyperplane $\mathcal{Z}(L) \subset \mathbb{P}^N_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, $\mathcal{Z}(L) \cap Z$ is not smooth.

   If you like, you may proceed as follows.

   (a) Show that $Z(\mathbb{F}_q) = \mathbb{P}^N(\mathbb{F}_q)$, i.e., that every point $[a_0, \cdots, a_n, b_0, \cdots, b_n]$ with each $a_i, b_j \in \mathbb{F}_q$ lies on Z. (Such a point corresponds to a closed point P of the scheme Z.)

   (b) Suppose that $a_0 = 1$, and let f be the dehomogenization $f = f(1, x_1, \cdots, x_n, y_0, \cdots, y_n)$. Compute the linearization $d_P(f)$. (HINT: *See 672HW9#4.*)

   (c) Let $\ell \in \mathbb{F}_q[x_1, \cdots, x_n, y_1, \cdots, y_n]$ be a linear polynomial. Show that there exists $P \in Z$ as above such that $T_P(Z) = \mathcal{Z}_{\mathbb{A}^N}(\ell)$.

   (d) Finish the claim.

6. (a) Suppose $f(x) \in K[x]$ is nonconstant. Show that $\mathcal{Z}_{\mathbb{A}^1}(f)$ is smooth if and only if $f(x)$ is squarefree. (HINT: *You may assume that K is algebraically closed (using, e.g., [GW]§6.12 and then show* $\mathcal{Z}_{\mathbb{A}^1}(f)$ *is regular.*)

   (b) Using Poonen's theorem, compute

$$\lim_{d\to\infty} \frac{\#\{f(x) \in \mathbb{F}_q[x] : \deg f \le d, f \text{ squarefree}\}}{\#\{f(x) \in \mathbb{F}_q[x] : \deg f \le d\}}.$$