Homework 1

Due: Friday, January 27

1. Describe all points of $\operatorname{Spec} \mathbb{Q}[x]$.

2. (a) Let $f = x^3 - 8 \in \mathbb{Z}[x]$. For each point $[\mathfrak{p}] \in \operatorname{Spec} \mathbb{Z}[x]$ listed below, compute the residue field $\kappa([\mathfrak{p}])$, and evaluate $f([\mathfrak{p}])$.
   i. $[(3)]$
   ii. $[(x-1)]$
   iii. $[(x^2+1)]$
   iv. $[(x^2+1, 3)]$

   (b) Find a ring A and a function $f \in A$ such that f is zero at every point of $\operatorname{Spec} A$, but A is not the zero function. (HINT: *See HW11#1 from last semester.*)

3. (a) Find rings A and B such that $A \not\cong B$ but $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are homeomorphic (as topological spaces).

   (b) Find rings A and B such that $A \not\cong B$, $|\operatorname{Spec} A|$ is in bijection with $|\operatorname{Spec} S|$, but $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are not homeomorphic.

   (HINT: *For (a), you can arrange so that* $|\operatorname{Spec} A|$ *and* $|\operatorname{Spec} B|$ *each consist of a single point; for (b), there's an example in which each set has two points.*)

4. Suppose $[\mathfrak{p}] \in \operatorname{Spec} A$. Show that $[\mathfrak{p}]$ is closed if and only if $\mathfrak{p}$ is a maximal ideal in A.

5. Suppose X is an affine variety over the algebraically closed field k, with coordinate ring $k[X]$. Show that there is a bijection between:
   - Closed points of the affine scheme $\operatorname{Spec} k[X]$.
   - Points of the variety X.

Go read the blog entry *Mumford's treasure map*, available at

`http://www.neverendingbooks.org/mumfords-treasure-map`