Homework 9

Due: Friday, October 21

1. *Cremona, revisited* Consider the rational map

$$\mathbb{P}^2 \overset{\phi}{\dashrightarrow} \mathbb{P}^2$$

$$[a_0, a_1, a_2] \longmapsto [a_1a_2, a_0a_2, a_0a_1]$$

defined on $\mathbb{P}^2 - \{[1,0,0],[0,1,0],[0,0,1]\}$. Let X_0, X_1 and X_2 be coordinates on the first copy of $\mathbb{P}^2$, and let Y_0, Y_1 and Y_2 be coordinates on the second copy.

The closure in $\mathbb{P}^2 \times \mathbb{P}^2$ of the graph of ϕ is the correspondence $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$ given by

$$Z = \mathcal{Z}_{\mathbb{P}}(X_0Y_0 - X_1Y_1, X_1Y_1 - X_2Y_2).$$

Let $P = [a_0, a_1, a_2]$.

   (a) Suppose $a_0a_1a_2 \neq 0$. What is $Z[P]$?
   (b) Suppose $a_0 = 0$, $a_1a_2 \neq 0$. What is $Z[P]$?
   (c) Suppose $a_0 = a_1 = 0$, $a_2 \neq 0$. What is $Z[P]$?

2. Recall the curve from class $C = \mathcal{Z}_{\mathbb{A}}(y^2 - (x^3 + 4x^2))$. Show that $k[C]$ is not integrally closed in its field of fractions, $k(C)$. (HINT: *Consider* $\frac{y}{x}$.)

   Extra: What is the integral closure of $k[C]$ in $k(C)$?

3. The blowup of $\mathbb{A}^2$ at $(0,0)$ is $\mathcal{Z}(xV_1 - yV_0)$, where x and y are coordinates on $\mathbb{A}^2$ and V_0 and V_1 are coordinates on $\mathbb{P}^1$. Let π be the projection $\pi : \mathcal{Z}(xV_1 - yV_0) \to \mathbb{A}^2$.

   Let $D = \mathcal{Z}(x^2 - y^2 - x^4 - y^4) \subset \mathbb{A}^2$.

   (a) What is $\pi^{-1}(D - \{(0,0)\}) \subset \mathbb{A}^2 \times \mathbb{P}^1$?
   (b) What is $\pi^{-1}((0,0))$?
   (c) Let $\widetilde{D}$ be the closure of $\pi^{-1}(D - \{0\})$ in $\mathbb{A}^2 \times \mathbb{P}^1$. What is $\widetilde{D} \cap \pi^{-1}((0,0))$?

4. *Linearization* If $f \in k[x_1, \cdots, x_n]$, and $P = (a_1, \cdots, a_n) \in \mathbb{A}^n$, the *linearization of f at P is*

$$d_P(f) = f(P) + \sum_{i=1}^{n} (\frac{\partial}{\partial x_i} f)(P)(x_i - a_i).$$

Suppose that $I = (f_1, \cdots, f_r) \subset k[x_1, \cdots, x_n]$ and that $P \in \mathcal{Z}(I)$. Let $d_P(I)$ be the ideal generated by the linearizations of all elements of I:

$$d_P(I) = (\{d_P(f) : f \in I\})k[x_1, \cdots, x_n].$$

Show that $d_P(I) = (d_P(f_1), \cdots, d_P(f_r))$. (HINT: *If $f \in I$ and $g \in k[x_1, \cdots, x_n]$, what is $d_P(fg)$?*)