Homework 8

Due: Friday, October 14

1. Recall that the (degree three) Veronese embedding

$$\mathbb{P}^1 \xrightarrow{\nu = \nu_{1,3}} \mathbb{P}^3$$

$$[a, b] \longmapsto [a^3, a^2b, ab^2, b^3]$$

has image $V_{1,3} = \mathcal{Z}_{\mathbb{P}}(F_0, F_1, F_2)$, where

$$\begin{aligned} F_0 &= Y_0Y_2 - Y_1^2 \\ F_1 &= Y_0Y_3 - Y_1Y_2 \\ F_2 &= Y_1Y_3 - Y_2^2 \end{aligned}$$

Show that for each $0 \leq i < j \leq 2$, $\mathcal{Z}_{\mathbb{P}}(F_i, F_j) \supsetneq V_{1,3}$, but all irreducible components of $\mathcal{Z}_{\mathbb{P}}(F_i, F_j)$ have dimension 1.

This shows that imposing an extra equation need not force the dimension to drop.

2. *Elimination theory, revisited* Suppose $Z = \mathcal{Z}_{\mathbb{P}^m \times \mathbb{A}^n}(I)$, where $I \subset k[X_0, \cdots, X_m, y_1, \cdots, y_n]$ is homogeneous in the X_i's.

The projective elimination ideal is

$$\hat{I} = \left\{ h \in k[y_1, \cdots, y_n] : \exists d > 0 : (X_0, \cdots, X_n)^d h \subset I \right\}.$$

Suppose $[a_0, \cdots, a_m] \times (b_1, \cdots, b_m) \in Z$ and $h \in \hat{I}$. Show that $h(b_1, \cdots, b_m) = 0$.

This shows $p_2(Z) \subseteq \mathcal{Z}_{\mathbb{A}^n}(\hat{I})$. In fact, one can show that $p_2(Z) = \mathcal{Z}_{\mathbb{A}^n}(\hat{I})$, which provides a proof that the map $p_2 : \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ is closed.

3. Consider the morphism

$$\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^2$$

$$[a, b] \longmapsto [a^3, a^2b + ab^2, b^3]$$

This induces a map of homogeneous coordinate rings

$$k_h[\mathbb{P}^2] \longrightarrow k_h[\mathbb{P}^1]$$

$$X \longmapsto S^3$$

$$Y \longmapsto S^2T + ST^2$$

$$Z \longmapsto T^3$$

(a) Describe the bihomogeneous ideal $I \subset k[S, T, X, Y, Z]$ such that $\mathcal{Z}(I) = \Gamma_\phi$, the graph of ϕ.

(b) Explain, in the notation of the previous problem, how to compute $\mathcal{I}_{\mathbb{P}^2}(\phi(\mathbb{P}^1))$, the ideal which defines the image of ϕ.

Extra: Using a computer algebra package if you like, compute $\mathcal{I}_{\mathbb{P}^2}(\phi(\mathbb{P}^1))$.

4. (a) Suppose X is a projective irreducible group variety. Prove that the group law on X is commutative.

   (b) A *ring variety* is a variety X equipped with

   addition A morphism $\alpha : X \times X \to X$;
   multiplication A morphism $\mu : X \times X \to X$;
   additive inverse A morphism $\iota : X \to X$;
   additive identity An element $\zeta \in X$

   such that (X, α, ζ) is a group variety, and multiplication satisfies the obvious axioms.

   Suppose that X is a projective irreducible ring variety. Prove that the multiplication map must be trivial.

   You may, and should, use basic properties of rings.

5. In class, we will prove a very geometric version of Noether's normalization lemma:

   Theorem *Suppose $X \subset \mathbb{P}^n$ is an irreducible subvariety,* $\dim X = r$. *Then there is a linear subspace $E \subset \mathbb{P}^n$ of dimension $n - r - 1$ such that $E \cap X = \emptyset$. Fix any such E. The projection π_E yields a finite-to-one map*

$$X \xrightarrow{\pi_E} \mathbb{P}^r$$

   The homogeneous coordinate ring $k_h[X] = k[X_0, \cdots, X_n]/I(X)$ is a finitely generated module over $k[Y_0, \cdots, Y_r] = k_h[E]$.

   Explain how to deduce the version usually stated in commutative algebra books:

   Let R be a finitely generated algebra over an algebraically closed field k. Then there exist elements $y_1, \cdots, y_r \in R$, algebraically independent over k, such that R is finite as a $k[y_1, \cdots, y_r]$-module.

   (Actually, this is true for an arbitrary, not-necessarily-algebraically-closed base field k.)