Homework 7

Due: Friday, October 7

1. *Veronese embeddings (I)*

   Consider the map $\nu = \nu_{1,d}$ given by

   $$\mathbb{P}^1 \xrightarrow{\quad \nu \quad} \mathbb{P}^d$$

   $$[a_0, a_1] \longmapsto [a_0^d, a_0^{d-1}a_1, \cdots, a_0 a_1^{d-1}, a_1^d].$$

   (a) Show that ν really is a morphism.

   (b) Prove that it is injective.

2. *Veronese (II)* Let $I = I_{1,d} \subset K[Y_0, \cdots, Y_d]$ be the homogeneous ideal generated by quadrics:

   $$I = \langle Y_i Y_j - Y_k Y_\ell : 0 \leq i, j, k, \ell \leq d, i + j = k + l \rangle.$$

   Let $V = V_{1,d}$ be the projective vanishing locus

   $$V = \mathcal{Z}_{\mathbb{P}}(I).$$

   Show that $\nu(\mathbb{P}^m) = V$.

   In fact, for every m and d there is a similar inclusion $\mathbb{P}^m \hookrightarrow \mathbb{P}^{\binom{m+d}{d}}$; the image is a projective variety whose ideal is generated by quadrics.

3. If $X = \mathcal{Z}_{\mathbb{P}}(I) \subset \mathbb{P}^n$, its homogeneous coordinate ring is the graded ring $k_h[X] = k[X_0, \cdots, X_n]/I$. (If we forget about the grading, this is just the (usual) coordinate ring of the affine cone over X.)

   Suppose that X is irreducible. The homogeneous localization of $k_h[X]$ at (0) is

   $$k_h[X]_{(0)} = \left\{ \frac{f}{g} : f, g \in k_h[X], g \neq 0, f \text{ and } g \text{ homogeneous}, \deg f = \deg g \right\}.$$

   Explain how to build an isomorphism

   $$k(X) \longrightarrow k_h[X]_{(0)};$$

   you need not actually prove your map is an isomorphism. (HINT: *Remember, elements of $k(X)$ are represented by (U, ϕ), where U is a nonempty open subset and ϕ is a regular function on U.*)

4. In spite of its name, the homogeneous coordinate ring depends not just on the variety, but on the embedding. Show that

   $$k_h[\mathbb{P}^1] \not\cong k_h[V_{1,2}].$$

   If you like, let $\mathfrak{m} = (Y_0, Y_1, Y_2)k_h[V_{1,2}]$. What is $\dim_k \mathfrak{m}/\mathfrak{m}^2$? Show that no maximal ideal of $k_h[\mathbb{P}^1]$ has this property.