Homework 6
Due: Friday, September 30

If $\phi : X \dashrightarrow Y$, the domain of ϕ $\operatorname{dom}(\phi)$ is the largest subset where ϕ is represented by a morphism.

1. (a) Let R be a ring, and let $f, g \in R[t]$ be polynomials. Suppose that

$$f(x) = g(1/x).$$

Show that $\deg f = \deg = 0$, and thus $f = g \in R$. *If $R = k$, this finishes the proof from class that any regular function on $\mathbb{P}^1$ is constant.*

(b) Suppose ϕ is a regular function on $\mathbb{P}^n$. Show that ϕ is a constant. (Hint: *On $U_i \cap U_j$, represent ϕ as*

$$\begin{aligned} \phi &= f(X_0/X_i, \cdots, X_n/X_i) \\ &= g(X_0/X_j, \cdots, X_n/X_i); \end{aligned}$$

view f as a polynomial in the variable X_j/X_i with coefficients in the field

$$R = k(X_0, \cdots, \widehat{X_i}, \widehat{X_j}, \cdots, X_n),$$

(omit X_i and X_j), and view g as a polynomial in X_j/X_i with coefficients in the same ring.)

2. Prove that a rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is actually regular.

3. (a) Let ϕ be the rational function on $\mathbb{P}^2$ given by $\phi = X_1/X_0$. What is $\operatorname{dom}(\phi)$?

(b) Now, think of this function as a rational map from $\mathbb{P}^2$. Compose this with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $t \mapsto [t, 1]$, to get a rational map $\overline{\phi} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. What is $\operatorname{dom}(\overline{\phi})$?

4. *Cremona transformations* Consider the rational map

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^2 \\ [a_0, a_1, a_2] & \longmapsto & [a_1a_2, a_0a_2, a_0a_1] \end{array}$$

(a) What is $\operatorname{dom}(\phi)$?

(b) Show that ϕ is birational, and is its own inverse.

(c) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\phi|_U : U \to V$ is an isomorphism.