Homework 4

Due: Friday, September 16

1. Let $X \subset \mathbb{A}^n$ be an algebraic set. Show that the following are equivalent:
   (a) X is irreducible;
   (b) $\mathcal{I}(X) \subset k[x_1, \cdots, x_n]$ is a prime ideal;
   (c) $k[X]$ is an integral domain.

2. (a) Let
   $$\Delta_{\mathbb{A}^n} = \{(P,P) : P \in \mathbb{A}^n\} \subset \mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}.$$
   Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be coordinates on $\mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$. What is $\mathcal{I}(\Delta_{\mathbb{A}^n})$? Show that $\Delta_{\mathbb{A}^n}$ is closed.

   (b) Suppose $V \subset \mathbb{A}^n$ is closed. Show that
   $$\Delta_V := \{(P,P); P \in V\} \subset \mathbb{A}^n \times \mathbb{A}^n$$
   is closed. (HINT: *$V \times V \subset \mathbb{A}^{2n}$ is closed.*)

3. Let $\phi : V \to W$ be a morphism.
   (a) The graph of ϕ is
   $$\Gamma_\phi := \{(P, \phi(P)) : P \in V\} \subset V \times W.$$
   Show that Γ_ϕ is closed. (HINT: *Consider the inverse image of Δ_W under $(\phi \times \mathrm{id}_W) : V \times W \to W \times W$.*)

   (b) Let $\psi : V \to W$ be a morphism. Show that $\{x \in V : \phi(x) = \psi(x)\}$ is closed.

4. Let V and W be k-vector spaces of dimensions m and n, respectively. After choosing a basis on V and W, we may identify $\mathbb{A}^{mn}$ with (the set of $m \times n$ matrices with entries in k, and thus with) $\mathrm{LinMap}(V, W)$.
   (a) Suppose $\dim V = \dim W$. Prove that the set of elements of $\mathrm{LinMap}(V, W)$ which are actually isomorphisms is a Zariski open subset of $\mathrm{LinMap}(V, W)$.

   (b) Let r be a nonnegative integer. Show that the set
   $$M_r := \{\alpha \in \mathrm{LinMap}(V, W) : \dim(\alpha(V)) \leq r\}$$
   is a Zariski closed subset of $\mathrm{LinMap}(V, W)$. (Here, *dim* means dimension as vector space.)

5. If X is a topological space, the topological dimension of X, $\mathrm{tdim}(X)$, is the supremum of the lengths of all chains

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n,$$

where each Z_i is a closed, irreducible subset of X.

If R is a ring, the height of a prime ideal $\mathfrak{p} \subset R$ is the supremum of the lengths of all chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \mathfrak{p}_n = \mathfrak{p}$$

of distinct prime ideals. The Krull dimension of R, $\mathrm{kdim}\, R$, is the supremum of the heights of all prime ideals.

Let Y be an irreducible affine variety.

(a) Prove that $\mathrm{tdim}(Y) = \mathrm{kdim}(k[Y])$.

(b) Use the following result from commutative algebra to show that $\mathrm{tdim}(Y) = \dim(Y)$.

Theorem Let R be an integral domain which is finitely generated as a k-algebra, and let K be the fraction field of R. Then

$$\mathrm{kdim}\, R = \mathrm{trdeg}(K/k);$$

the Krull dimension of R is the transcendence degree of K over k.