Homework 3

Due: Friday, September 9

1. If $f \in k[x_1, \cdots, x_n]$, the associated distinguished[1] affine open set is

$$D(f) := \{P \in \mathbb{A}^n : f(P) \neq 0\}.$$

   (a) Suppose $f, g \in k[x_1, \cdots, x_n]$. Show that $D(fg) = D(f) \cap D(g)$.

   (b) Show that the collection of distinguished open sets in $\mathbb{A}^n$ is a basis for the Zariski topology on $\mathbb{A}^n$.

   Recall that if X is a topological space, then a collection of open subsets $\mathcal{C}$ is a basis for the topology on X if for every open set U of X, and each $x \in U$, there is some $V \in \mathcal{C}$ such that

$$x \in V \subseteq U.$$

2. (a) A topological space X is called quasicompact if every open cover admits a finite subcover.

   Suppose $\mathcal{C}$ is a basis for the topology of X. Prove that X is quasicompact if and only if every open covering $X = \cup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{C}$ admits a finite subcover.

   (b) Prove that $\mathbb{A}^n$ is quasicompact.

3. Let $H = \mathcal{Z}(x_1x_2 - 1) \subset \mathbb{A}^2$, and let $\mathbb{A}^1$ be the affine line with coordinate t, so that $k[\mathbb{A}^1] = k[t]$.

   Consider the morphism

$$\begin{array}{ccc} H & \xrightarrow{\phi} & \mathbb{A}^1 \\ (a_1, a_2) & \longmapsto & a_1 \end{array}$$

   (a) Describe the map $\phi^* : k[\mathbb{A}^1] \to k[H]$.

   (b) Is ϕ^* injective? Surjective?

   (c) Describe the image of ϕ. Is it a closed subset of $\mathbb{A}^1$?

4. Consider the morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{\beta} & \mathbb{A}^2 \\ (a_1, a_2) & \longmapsto & (a_1, a_1a_2) \end{array}$$

[1] or principal, or standard, or basic

(a) Is β injective? Explain.

(b) Is β surjective? Explain.

(c) Describe open subsets $U \subset \mathbb{A}^2$ and $V \subset \mathbb{A}^2$ such that β gives an isomorphism $\beta|_U : U \to V$.

5. (a) Let $W \subset \mathbb{A}^n$ be closed, and suppose that $U \subset W$. Show that $\mathcal{I}(U) = \mathcal{I}(W)$ if and only if U is dense in W.

   (b) Let $\phi : V \to W$ be a morphism. Show that the image of ϕ is dense in W if and only if $\phi^* : k[W] \to k[V]$ is injective. *Such a morphism is called* dominant.