Homework 2

Due: Friday, September 2

Material for problems 3, 4 and 5 will be covered in class the week of August 29.

1. Let $\phi : R \to S$ be a ring homomorphism, and let $J \subset S$ be an ideal. Let $I = \phi^{-1}(J)$.
   (a) Show that I is an ideal of R.
   (b) Show that if J is prime, then I is prime.
   (c) Give an example to show that even if J is maximal, I need not be maximal.
2. Suppose that $R = k[x_1, \cdots, x_m]/\mathfrak{a}$ and $S = k[y_1, \cdots, y_n]/\mathfrak{b}$, where k is algebraically closed. Let $\phi : R \to S$ be a ring homomorphism. Show that if $J \subset S$ is maximal, then $\phi^{-1}(J)$ is maximal.

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3. Let k be an algebraically closed field, and suppose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$. Show that there is no common solution $f_1 = f_2 = \cdots = f_r = 0$ if and only if there are $a_1, \cdots, a_r \in k[x_1, \cdots, x_n]$ such that
$$\sum_{i=1}^{r} a_i f_i = 1.$$
4. (a) Find polynomials
$$a(x) = \sum_{j=0}^{4} a_i x^i \text{ and } b(x) = \sum_{j=0}^{4} b_i x^i$$
such that
$$a(x) \cdot (x^2 + 1) + b(x) \cdot (x^3 + 1) = 1.$$
(HINT: *Solve for a_i and b_i.*)

   (b) Suppose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$ have no common zero. Suppose you know there is an N such that there are polynomials $g_1, \cdots, g_r \in k[x_1, \cdots, x_n]$ such that $\deg f_i g_i \leq N$ and
$$\sum f_i g_i = 1.$$
Explain (briefly) how you would use linear algebra to find such polynomials.

   An *effective nullstellensatz* gives a computable value of N in terms of n, r, and the degree $f_1, \cdots, f_r$. See, e.g., J. Kollár, *Sharp effective Nullstellensatz*, JAMS 1 (1988), 963-765; and Z. Jelonek, *On the effective Nullstellensatz*, Inv. Math. 162 (2005), 1–17.
5. There is a natural identification (of sets) $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^2$. Show that the Zariski topology on $\mathbb{A}^2$ is strictly finer than the product topology of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

   Concretely, show:

(a) Suppose $C_1, \cdots, C_r$ and $D_1, \cdots, D_r$ are closed subsets of $\mathbb{A}^1$. Then

$$\cup_{i=1}^{r} C_i \times D_i \subset \mathbb{A}^2 \tag{1}$$

is closed.

(b) Find a set $S \subset \mathbb{A}^2$ which is closed but is *not* of the form (1).