Homework 12
Due: Friday, November 11

1. *Finiteness, revisited.*

   A weak version of the Cohen-Seidenberg going-up theorem says:

   Theorem Let $B \subseteq A$ be rings, A integral over B, and let $\mathfrak{q}$ be a prime ideal of B. Then there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap B = \mathfrak{q}$.

   Use this to show that a finite dominant map between irreducible quasiprojective varieties is surjective.

2. *We have implicitly been using something like this in class.*

   (a) Suppose $g(z) \in k[z]$ is a polynomial. Show (directly) that $(z-\alpha)$ is a multiple root of $g(z)$ if and only if $(z-\alpha)|g(z)$ and $(z-\alpha)|g'(z)$. Show that $g(z)$ has repeated roots if and only if $\gcd(g(z), g'(z)) \neq 1$.

   (b) Suppose Y is a normal affine variety, and that X is a normal variety equipped with a finite map $\phi : X \to Y$ such that $k[X] \cong k[Y]/(f(z))$, where $f(z) = z^d + \sum_{i=0}^{d-1} a_i z^i$ with $a_i \in k[Y]$.

   Suppose $Q \in Y$, and let $f_Q(z) = z^d + \sum_i a_i(Q) z^i \in k[z]$. Show that $\#\phi^{-1}(Q) = d$ if and only if $\gcd(f_Q(z), f_Q'(z)) = 1$. *This is equivalent to the condition ϕ is smooth on $\phi^{-1}(Q)$.*

We will make extensive use of the dual space in our proof(s) of Bézout's theorem.

3. The *dual space* of $\mathbb{P}^2$ is $\mathbb{P}^{2*}$, the space of lines in $\mathbb{P}^2$. In fact, $\mathbb{P}^{2*}$ is isomorphic to $\mathbb{P}^2$; a point $[a_0, a_1, a_2] \in \mathbb{P}^{2*}$ corresponds $\mathcal{Z}_{\mathbb{P}}(a_0X_0 + a_1X_1 + a_2X_2)$. (Note that is well-defined on equivalence classes!)

   Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous form, and let $X = \mathcal{Z}(F)$ be the associated plane curve, with smooth locus X^{sm}.

   Show that the map

$$\begin{array}{ccc} X^{\mathrm{sm}} & \xrightarrow{\phi} & \mathbb{P}^{2*} \\ P & \longmapsto & T_P X \end{array}$$

   (where $T_P X$ is the closure of the *external* tangent space to X at P) is a morphism, by giving an explicit formula for ϕ in terms of F and the coordinates on $\mathbb{P}^2$.

The closure of the image is called the dual curve X^*.

4. Continue to assume $X = \mathcal{Z}(F) \subset \mathbb{P}^2$.

   (a) Show that the set of $L \in \mathbb{P}^{2*}$ which pass through a singular point of X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (b) Show that the set of $L \in \mathbb{P}^{2*}$ which are tangent to X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (c) Suppose $\deg F = d$. Show that there is an open subset $U \subset \mathbb{P}^{2*}$ such that for each $L \in U$, $L \cap X$ consists of exactly d points.