Homework 10

Due: Friday, November 4

1. The ring of *dual numbers* is $k[\epsilon] = k[t]/(t^2)$, where ϵ is the coset $t + (t^2)$. Note that as a vector space, $k[\epsilon] = \{a_0 + a_1\epsilon : a_0, a_1 \in k\}$. *Make sure you understand how multiplication works in this ring.*

   (a) Let R be a k-algebra. Suppose $\alpha : R \to k[\epsilon]$ is a ring homomorphism; write it as

   $$\begin{array}{ccc} R & \xrightarrow{f} & k[\epsilon] \\ \\ f & \longmapsto & \alpha_0(f) + \epsilon\alpha_1(f) \end{array}$$

   Define a map of k-modules

   $$\begin{array}{ccc} R & \xrightarrow{D_\alpha} & \epsilon k[\epsilon] \\ \\ f & \longmapsto & \alpha(f) - \alpha_0(f) \\ \\ & & = \epsilon\alpha_1(f) \end{array}$$

   Show that D_α is a k-linear derivation.
   By composing this with the isomorphism of k-modules

   $$\begin{array}{ccc} \epsilon k[\epsilon] & \longrightarrow & k \\ \\ \epsilon a_1 & \longmapsto & a_1 \end{array}$$

   we may view D_α as an element of $\mathrm{Der}_k(R, k)$.

   (b) Conversely, given $D \in \mathrm{Der}_k(R, k)$, explain how to find $\alpha \in \mathrm{Hom}(R, k[\epsilon])$ such that $D = D_\alpha$.

   (c) Suppose X is an affine variety, and $\alpha \in \mathrm{Hom}(k[X], k[\epsilon])$. Show that there is a point $P \in X$ such that D_α is a derivation centered at P. (HINT: *Compose with the map $k[\epsilon] \to k$, $a_0 + a_1\epsilon \mapsto a_0$.*)
   One can use this to produce an isomorphism between $T_P X$ and a certain subgroup of $\mathrm{Hom}(k[X], k[\epsilon])$.

2. Fix $n \in \mathbb{N}$. Recall that if R is any ring, then $\mathrm{GL}_n(R)$ is the subset of those elements of $\mathrm{Mat}_n(R)$ which admit multiplicative inverses.

   Consider $\mathrm{Mat}_n(k[\epsilon])$; note that any $M \in \mathrm{Mat}_n(k[\epsilon])$ can be written as $M_0 + \epsilon M_1$, where $M_0, M_1 \in \mathrm{Mat}_n(k)$. Let I_n be the identity matrix.

(a) Suppose $A, B \in \mathrm{Mat}_n(k[\epsilon])$. Show that

$$(I_n + \epsilon A)(I_n + \epsilon B) = I_n + \epsilon(A + B).$$

(b) Show that for any $A \in \mathrm{Mat}_n(k)$, $I_n + \epsilon A \in \mathrm{GL}_n(k[\epsilon])$. (HINT: *Use (a).*)

(c) Describe those $A \in \mathrm{Mat}_2(k)$ for which $I_2 + \epsilon A \in \mathrm{SL}_2(k[\epsilon])$.

In general, this gives a way of calculating the Lie algebra of an algebraic group, i.e., the tangent space at the identity of that group.

3. If X is a variety, and P is a smooth point of X, a system of local parameters at P is a collection of elements $f_1, \cdots, f_r \in \mathfrak{M}_P \subset \mathcal{O}_{X,P}$ such that $f_1 \bmod \mathfrak{M}_P^2, \cdots, f_r \bmod \mathfrak{M}_p^2$ is a basis for the k-vector space $\mathfrak{M}_P / \mathfrak{M}_P^2$.

   (a) Look up Nakayama's lemma; write down the version you find, and use it to show that if $f_1, \cdots, f_r$ is a system of local parameters, then $(f_1, \cdots, f_r) = \mathfrak{M}_P$.

   (b) Prove that a polynomial $f \in k[T] = k[\mathbb{A}^1]$ is a local parameter at the point $P : T = \alpha$ if and only if α is a simple root of f.

The next two questions don't use anything recent, but taken together give an independent proof of a special case of something we'll do in class.

4. Suppose $X \subset \mathbb{P}^n$ is a projective irreducible variety. Let H be homogeneous of degree d, and let Y be the hypersurface $Y = \mathcal{Z}_{\mathbb{P}}(H)$.

   (a) Suppose $X \cap Y = \emptyset$. Let F be any homogeneous form of degree d. Show that the function on X given by $P \mapsto F(P)/H(P)$ is constant. (HINT: *What do you know about regular functions on a projective variety?*)

   (b) Continue to suppose $X \cap Y = \emptyset$. Show that for any homogeneous polynomials F and G of degree d, where $G \notin \mathcal{I}_{\mathbb{P}}(X)$, the rational function on X $P \mapsto F(P)/G(P)$ is constant.

5. Suppose $X \subset \mathbb{P}^n$ is a projective variety of positive dimension. Let $Y \subset \mathbb{P}^n$ be a hypersurface. Show that $X \cap Y$ is nonempty.