Homework 10
Due: Friday, October 28

1. *Please prove directly this special case of a theorem from class.* Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. Recall that X is singular at P if $\dim T_P X > \dim(X)$, and is smooth or nonsingular at P if $\dim T_P X = \dim(X)$.

   Suppose $X = \mathcal{Z}(f)$ is an irreducible hypersurface.

   (a) Suppose $P \in X$. Show that X is singular at P if and only if $(\frac{\partial}{\partial x_i} f)(P) = 0$ for each $i = 1, \cdots, n$.

   (b) Show that the set of smooth points of X is a non-empty open subset.

2. Let $f(x,y) \in k[x,y]$ be a non-constant polynomial, and let $X = \mathcal{Z}(f)$. If $P = (a,b) \in \mathbb{A}^2$, expand f in a Taylor series centered at P:

$$\begin{aligned} f &= f(P) + (\frac{\partial}{\partial x} f(P)(x-a) + \frac{\partial}{\partial y} f(P)(y-b)) + \\ &\quad \frac{1}{2}(\frac{\partial^2}{\partial x^2} f(P)(x-a)^2 + 2\frac{\partial^2}{\partial x \partial y} f(P)(x-a)(y-b) + \frac{\partial^2}{\partial y^2} f(P)(y-b)^2) + \cdots \\ &= \sum f_i \end{aligned}$$

   where $f_i \in k[x,y]$ is homogeneous of degree i (in the grading which gives degree one to $(x-a)$ and $(y-b)$).

   The multiplicity of X at P, $\mu_P(X)$, is the smallest r such that $f_r \neq 0$.

   (a) Show that $P \in X \iff \mu_P(X) \geq 1$.

   (b) Show that X is singular at $P \iff \mu_P(X) \geq 2$.

3. Let $f = y^4 - 2y^3 + y^2 - 3x^2y + 2x^4$, and let $X = \mathcal{Z}(f) \subset \mathbb{A}^2$.

   (a) Find all singular points of X.

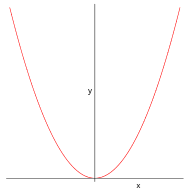
   (b) For each singular point P, compute $\mu_P(X)$.

4. As in #2, let $f(x,y) \in k[x,y]$ be nonconstant, $P \in \mathcal{Z}(f)$, $f = \sum f_i$ in coordinates centered at P.

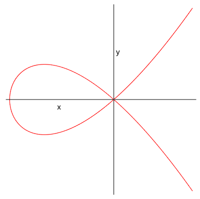
   The *initial term*, or leading term, of f at P, $\mathrm{In}_P(f)$, is the nonzero term f_i of smallest degree. The *tangent cone* of X at P is $\mathcal{Z}(\mathrm{In}_P(f))$.

   For each of the following planar curves C, compute the tangent cone of C at $P = (0,0)$. Graph the real points of this tangent cone.

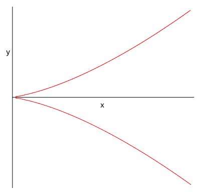
(a) $\alpha(x, y) = y - x^2$.



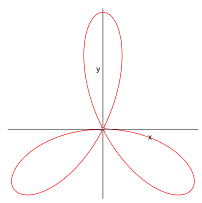
(b) $\beta(x, y) = y^2 - x^3 - x^2$



(c) $\gamma(x, y) = y^2 - x^3$



(d) $\delta(x, y) = (x^2 + y^2)^2 + 3x^2y - y^3$



5. *You may have already done this in the service of HW 5#1.*

   Suppose $F \in k[x, y]$ is homogeneous of degree $d \geq 1$. Show that there is a factorization

$$F(x, y) = x^{e_0} \prod_{j=1}^{r} (y - a_i x)^{e_i}$$

   where $a_i \neq a_j$ for $i \neq j$.

   If $F = \mathrm{In}_{(0,0)}(f)$, then $\deg F$ is the multiplicity of $\mathcal{Z}(f)$ at the origin, and e_i is the multiplicity of $\mathcal{Z}(f)$ along the line $\mathcal{Z}(y - a_i x)$.