1. Please prove directly this special case of a theorem from class. Let \( X \subset \mathbb{A}^n \) be an affine variety, and let \( P \in X \) be a point. Recall that \( X \) is singular at \( P \) if \( \dim T_P X > \dim(X) \), and is smooth or nonsingular at \( P \) if \( \dim T_P X = \dim(X) \).

Suppose \( X = Z(f) \) is an irreducible hypersurface.

(a) Suppose \( P \in X \). Show that \( X \) is singular at \( P \) if and only if \( (\frac{\partial}{\partial x_i} f)(P) = 0 \) for each \( i = 1, \ldots, n \).

(b) Show that the set of smooth points of \( X \) is a non-empty open subset.

2. Let \( f(x, y) \in k[x, y] \) be a non-constant polynomial, and let \( X = Z(f) \). If \( P = (a, b) \in \mathbb{A}^2 \), expand \( f \) in a Taylor series centered at \( P \):

\[
f = f(P) + \left( \frac{\partial}{\partial x} f(P)(x - a) + \frac{\partial}{\partial y} f(P)(y - b) \right) + \\
\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} f(P)(x - a)^2 + 2 \frac{\partial^2}{\partial x \partial y} f(P)(x - a)(y - b) + \frac{\partial^2}{\partial y^2} f(P)(y - b)^2 \right) + \cdots \\
= \sum f_i
\]

where \( f_i \in k[x, y] \) is homogeneous of degree \( i \) (in the grading which gives degree one to \((x - a)\) and \((y - b)\)).

The multiplicity of \( X \) at \( P \), \( \mu_P(X) \), is the smallest \( r \) such that \( f_r \neq 0 \).

(a) Show that \( P \in X \iff \mu_P(X) \geq 1 \).

(b) Show that \( X \) is singular at \( P \iff \mu_P(X) \geq 2 \).

3. Let \( f = y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 \), and let \( X = Z(f) \subset \mathbb{A}^2 \).

(a) Find all singular points of \( X \).

(b) For each singular point \( P \), compute \( \mu_P(X) \).

4. As in #2, let \( f(x, y) \in k[x, y] \) be nonconstant, \( P \in Z(f) \), \( f = \sum f_i \) in coordinates centered at \( P \).

The initial term, or leading term, of \( f \) at \( P \), \( \text{In}_P(f) \), is the nonzero term \( f_i \) of smallest degree. The tangent cone of \( X \) at \( P \) is \( Z(\text{In}_P(f)) \).

For each of the following planar curves \( C \), compute the tangent cone of \( C \) at \( P = (0, 0) \). Graph the real points of this tangent cone.
(a) $\alpha(x, y) = y - x^2$.

(b) $\beta(x, y) = y^2 - x^3 - x^2$.

(c) $\gamma(x, y) = y^2 - x^3$.

(d) $\delta(x, y) = (x^2 + y^2)^2 + 3x^2y - y^3$.

5. You may have already done this in the service of HW 5#1.

Suppose $F \in k[x, y]$ is homogeneous of degree $d \geq 1$. Show that there is a factorization

$$F(x, y) = x^{e_0} \prod_{j=1}^{r} (y - a_i x)^{e_i}$$

where $a_i \neq a_j$ for $i \neq j$.

If $F = \text{In}_{(0,0)}(f)$, then $\deg F$ is the multiplicity of $\mathcal{Z}(f)$ at the origin, and $e_i$ is the multiplicity of $\mathcal{Z}(f)$ along the line $\mathcal{Z}(y - a_i x)$. 