1. Recall the curve from class $C = Z_A(y^2 - (x^3 + 4x^2))$. Show that $k[C]$ is not integrally closed in its field of fractions, $k(C)$. (HINT: Consider $\frac{y}{x}$.)

**Extra:** What is the integral closure of $k[C]$ in $k(C)$?

2. The blowup of $\mathbb{A}^2$ at $(0, 0)$ is $Z(xU_1 - yU_0)$, where $x$ and $y$ are coordinates on $\mathbb{A}^2$ and $U_0$ and $U_1$ are coordinates on $\mathbb{P}^1$. Let $\pi$ be the projection $\pi : Z(xU_1 - yU_0) \to \mathbb{A}^2$.

Let $Y = Z(x^2 - y^2 - x^4 - y^4) \subset \mathbb{A}^2$.

(a) What is $\pi^{-1}(Y - \{(0, 0)\}) \subset \mathbb{A}^2 \times \mathbb{P}^1$?

(b) What is $\pi^{-1}((0, 0))$?

(c) Let $\widetilde{Y}$ be the closure of $\pi^{-1}(Y - \{0\})$ in $\mathbb{A}^2 \times \mathbb{P}^1$. What is $\widetilde{Y} \cap \pi^{-1}((0, 0))$?

3. Cremona, revisited Consider the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$[a_0, a_1, a_2] \mapsto [a_1a_2, a_0a_2, a_0a_1]$$

defined on $\mathbb{P}^2 - \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$. Let $X_0$, $X_1$ and $X_2$ be coordinates on the first copy of $\mathbb{P}^2$, and let $Y_0$, $Y_1$ and $Y_2$ be coordinates on the second copy.

The closure in $\mathbb{P}^2 \times \mathbb{P}^2$ of the graph of $\phi$ is the correspondence $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$ given by

$$Z = Z_{\phi}(X_0 Y_0 - X_1 Y_1, X_1 Y_1 - X_2 Y_2).$$

Let $P = [a_0, a_1, a_2]$.

(a) Suppose $a_0a_1a_2 \neq 0$. What is $Z[P]$?

(b) Suppose $a_0 = 0$, $a_1a_2 \neq 0$. What is $Z[P]$?

(c) Suppose $a_0 = a_1 = 0$, $a_2 \neq 0$. What is $Z[P]$?

4. Let $W$, $X$ and $Y$ be irreducible projective varieties. Suppose $Z_{WX} \subset W \times X$ and $Z_{XY} \subset X \times Y$ are correspondences. Explain how to “compose” these to obtain a correspondence $Z \subset W \times Y$. If $P \in W$, what is $Z[P]$?

5. Linearization If $f \in k[x_1, \cdots, x_n]$, and $P = (a_1, \cdots, a_n) \in \mathbb{A}^n$, the linearization of $f$ at $P$ is

$$d_P(f) = f(P) + \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} f \right)(P)(x_i - a_i).$$
Suppose that \( I = (f_1, \cdots, f_r) \subset k[x_1, \cdots, x_n] \) and that \( P \in Z(I) \). Let \( d_P(I) \) be the ideal generated by the linearizations of all elements of \( I \):
\[
d_P(I) = (\{d_P(f) : f \in I\})k[x_1, \cdots, x_n].
\]

Show that \( d_P(I) = (d_P(f_1), \cdots, d_P(f_r)) \). (HINT: If \( f \in I \) and \( g \in k[x_1, \cdots, x_n] \), what is \( d_P(fg) \)?)