Homework 6
Due: Oct 15

These first two should probably have been assigned earlier.

1. If X is a topological space, the topological dimension of X, $\mathrm{tdim}(X)$, is the supremum of the lengths of all chains
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n,$$
where each Z_i is a closed, irreducible subset of X.

   If R is a ring, the height of a prime ideal $\mathfrak{p} \subset R$ is the supremum of the lengths of all chains
$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \mathfrak{p}_n = \mathfrak{p}$$
of distinct prime ideals. The Krull dimension of R, $\mathrm{kdim}\, R$, is the supremum of the heights of all prime ideals.

   Let Y be an irreducible affine variety.

   (a) Prove that $\mathrm{tdim}(Y) = \mathrm{kdim}(k[Y])$.

   (b) Use the following result from commutative algebra to show that $\mathrm{tdim}(Y) = \dim(Y)$.

   Theorem Let R be an integral domain which is finitely generated as a k-algebra. Then $\mathrm{kdim}\, R = \mathrm{tr.\,deg.}(\mathrm{Frac}(R)/k)$.

2. Let V and W be k-vector spaces of dimensions m and n, respectively. After choosing a basis on V and W, we may identify $\mathbb{A}^{mn}$ with (the set of $m \times n$ matrices with entries in k, and thus with) $\mathrm{LinMap}(V, W)$.

   (a) Suppose $\dim V = \dim W$. Prove that the set of elements of $\mathrm{LinMap}(V, W)$ which are actually isomorphisms is a Zariski open subset of $\mathrm{LinMap}(V, W)$.

   (b) Let r be a nonnegative integer. Show that the set
$$M_r := \{\alpha \in \mathrm{LinMap}(V, W) : \dim(\alpha(V)) \le r\}$$
is a Zariski closed subset of $\mathrm{LinMap}(V, W)$. (Here, *dim* means dimension as vector space.)

3. A *ring variety* is a variety X equipped with

   addition A morphism $\alpha : X \times X \to X$;

   multiplication A morphism $\mu : X \times X \to X$;

   additive inverse A morphism $\iota : X \to X$;

   additive identity An element $\zeta \in X$

such that (X, α, ζ) is a group variety, and multiplication satisfies the obvious axioms.

Suppose that X is a projective irreducible ring variety. Prove that the multiplication map must be trivial.

You may, and should, use basic properties of rings.

4. *Elimination theory, revisited* Suppose $Z = \mathcal{Z}_{\mathbb{P}^m \times \mathbb{A}^n}(I)$, where $I \subset k[X_0, \cdots, X_m, y_1, \cdots, y_n]$ is homogeneous in the X_i's.

   The projective elimination ideal is

   $$\hat{I} = \{h \in k[y_1, \cdots, y_n] : \exists d > 0 : (X_0, \cdots, X_n)^d h \subset I\}.$$

   Suppose $[a_0, \cdots, a_m] \times (b_1, \cdots, b_m) \in Z$ and $h \in \hat{I}$. Show that $h(b_1, \cdots, b_m) = 0$.

   This shows $p_2(Z) \subseteq \mathcal{Z}_{\mathbb{A}^n}(\hat{I})$. In fact, one can show that $p_2(Z) = \mathcal{Z}_{\mathbb{A}^n}(\hat{I})$, which provides a proof that the map $p_2 : \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ is closed.

5. Consider the morphism

   $$\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^2$$

   $$[a, b] \longmapsto [a^3, a^2b + ab^2, b^3]$$

   This induces a map of homogeneous coordinate rings

   $$k_h[\mathbb{P}^2] \longrightarrow k_h[\mathbb{P}^1]$$

   $$X \longmapsto S^3$$

   $$Y \longmapsto S^2T + ST^2$$

   $$Z \longmapsto T^3$$

   (a) Describe the bihomogeneous ideal $I \subset k[S, T, X, Y, Z]$ such that $\mathcal{Z}(I) = \Gamma_\phi$, the graph of ϕ.

   (b) Explain, in the notation of the previous problem, how to compute $\mathcal{I}_{\mathbb{P}^2}(\phi(\mathbb{P}^1))$, the ideal which defines the image of ϕ.

Extra: Using a computer algebra package if you like, compute $\mathcal{I}_{\mathbb{P}^2}(\phi(\mathbb{P}^1))$.