Homework 6  
Due: Oct 15

These first two should probably have been assigned earlier.

1. If $X$ is a topological space, the topological dimension of $X$, $\text{tdim}(X)$, is the supremum of the lengths of all chains $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$, where each $Z_i$ is a closed, irreducible subset of $X$.

If $R$ is a ring, the height of a prime ideal $p \subset R$ is the supremum of the lengths of all chains $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = p$ of distinct prime ideals. The Krull dimension of $R$, $\text{kdim} R$, is the supremum of the heights of all prime ideals.

Let $Y$ be an irreducible affine variety.

(a) Prove that $\text{tdim}(Y) = \text{kdim}(k[Y])$.

(b) Use the following result from commutative algebra to show that $\text{tdim}(Y) = \text{dim}(Y)$.

**Theorem**  Let $R$ be an integral domain which is finitely generated as a $k$-algebra. Then $\text{kdim} R = \text{tr. deg.}(\text{Frac}(R)/k)$.

2. Let $V$ and $W$ be $k$-vector spaces of dimensions $m$ and $n$, respectively. After choosing a basis on $V$ and $W$, we may identify $A^{mn}$ with (the set of $m \times n$ matrices with entries in $k$, and thus with) $\text{LinMap}(V, W)$.

(a) Suppose $\text{dim} V = \text{dim} W$. Prove that the set of elements of $\text{LinMap}(V, W)$ which are actually isomorphisms is a Zariski open subset of $\text{LinMap}(V, W)$.

(b) Let $r$ be a nonnegative integer. Show that the set $M_r := \{ \alpha \in \text{LinMap}(V, W) : \text{dim}(\alpha(V)) \leq r \}$ is a Zariski closed subset of $\text{LinMap}(V, W)$. (Here, $\text{dim}$ means dimension as vector space.)

3. A **ring variety** is a variety $X$ equipped with

- **addition** A morphism $\alpha : X \times X \to X$;
- **multiplication** A morphism $\mu : X \times X \to X$;
- **additive inverse** A morphism $\iota : X \to X$;
- **additive identity** An element $\zeta \in X$
such that \((X, \alpha, \zeta)\) is a group variety, and multiplication satisfies the obvious axioms.

Suppose that \(X\) is a projective irreducible ring variety. Prove that the multiplication map must be trivial.

*You may, and should, use basic properties of rings.*

4. **Elimination theory, revisited** Suppose \(Z = Z_{\mathbb{P}^m \times \mathbb{A}^n}(I)\), where \(I \subset k[X_0, \cdots, X_m, y_1, \cdots, y_n]\) is homogeneous in the \(X_i\)'s.

The projective elimination ideal is

\[
\hat{I} = \{ h \in k[y_1, \cdots, y_n] : \exists d > 0 : (X_0, \cdots, X_n)^d h \subset I \}.
\]

Suppose \([a_0, \cdots, a_m] \times (b_1, \cdots, b_m) \in Z\) and \(h \in \hat{I}\). Show that \(h(b_1, \cdots, b_m) = 0\).

*This shows \(p_2(Z) \subseteq Z_{\mathbb{A}^n}(\hat{I})\). In fact, one can show that \(p_2(Z) = Z_{\mathbb{A}^n}(\hat{I})\), which provides a proof that the map \(p_2 : \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n\) is closed.*

5. Consider the morphism

\[
\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^2
\]

\([a, b] \longmapsto [a^3, a^2 b + ab^2, b^3]\)

This induces a map of homogeneous coordinate rings

\[
\begin{align*}
\mathbb{P}^2 & \longrightarrow k_h[\mathbb{P}^2] \\
X & \longmapsto S^3 \\
Y & \longmapsto S^2 T + ST^2 \\
Z & \longmapsto T^3
\end{align*}
\]

(a) Describe the bihomogeneous ideal \(I \subset k[S, T, X, Y, Z]\) such that \(Z(I) = \Gamma_\phi\), the graph of \(\phi\).

(b) Explain, in the notation of the previous problem, how to compute \(I_{\mathbb{P}^2}(\phi(\mathbb{P}^1))\), the ideal which defines the image of \(\phi\).

**Extra:** Using a computer algebra package if you like, compute \(I_{\mathbb{P}^2}(\phi(\mathbb{P}^1))\).