Homework 5

Due: Wednesday, ~~October 1~~ October 8

1. Prove that a rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is actually regular.

2. (a) Let ϕ be the rational function on $\mathbb{P}^2$ given by $\phi = X_1/X_0$. What is $\mathrm{dom}(\phi)$?

   (b) Now, think of this function as a rational map from $\mathbb{P}^2$. Compose this with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $t \mapsto [t, 1]$, to get a rational map $\overline{\phi} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. What is $\mathrm{dom}(\overline{\phi})$?

3. *Cremona transformations* Consider the rational map

$$\mathbb{P}^2 \overset{\phi}{\dashrightarrow} \mathbb{P}^2$$

$$[a_0, a_1, a_2] \longmapsto [a_1 a_2, a_0 a_2, a_0 a_1]$$

   (a) What is $\mathrm{dom}(\phi)$?

   (b) Show that ϕ is birational, and is its own inverse.

   (c) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\phi|_U : U \to V$ is an isomorphism.

4. *Veronese embeddings* Fix natural numbers m and d. The set of all homogeneous polynomials of degree d in the variables $X_0, \cdots, X_m$ is a vector space over k of dimension $\binom{m+d}{d}$. Let

$$\begin{aligned} E &:= \{\underline{e} = (e_0, \cdots, e_m) : \text{ each } e_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum e_i = d\} \\ f_{\underline{e}} &:= \underline{X}^{\underline{e}} \text{ for } \underline{e} \in E \\ &= X_0^{e_0} \cdots X_m^{e_m} \end{aligned}$$

   Then $\{f_{\underline{e}} : e \in E\}$ is a basis for the space of homogeneous polynomials of degree d.

   Let $N = \binom{m+d}{d} - 1$, and let $\{Z_{\underline{e}} : e \in E\}$ be coordinates on $\mathbb{P}^N$. Consider the rational map

$$\mathbb{P}^m \overset{\nu}{\dashrightarrow} \mathbb{P}^N$$

$$[a_0, \cdots, a_n] \longmapsto [\cdots, f_{\underline{e}}(a_0, \cdots, a_n), \cdots]$$

   (a) Show that ν is actually a morphism $\mathbb{P}^m \to \mathbb{P}^N$.

   (b) Show that ν is an inclusion.

5. *Continuation of 4* Let I be the ideal generated by the quadrics

$$\{Z_{\underline{e}^{(1)}} Z_{\underline{e}^{(2)}} - Z_{\underline{e}^{(3)}} Z_{\underline{e}^{(4)}} : \underline{e}^{(i)} \in E, \underline{e}^{(1)} + \underline{e}^{(2)} = \underline{e}^{(3)} + \underline{e}^{(4)} \in \mathbb{Z}^{m+1}\}$$

   Show that $\nu(\mathbb{P}^m) = \mathcal{Z}_{\mathbb{P}}(I)$, and that the inverse of ν is a morphism from $\mathcal{Z}_{\mathbb{P}}(I)$ to $\nu(\mathbb{P}^m)$.