1. Prove that a rational map \( \phi : \mathbb{P}^1 \to \mathbb{P}^n \) is actually regular.

2. (a) Let \( \phi \) be the rational function on \( \mathbb{P}^2 \) given by \( \phi = X_1/X_0 \). What is \( \text{dom}(\phi) \)?
   (b) Now, think of this function as a rational map from \( \mathbb{P}^2 \). Compose this with the inclusion \( \mathbb{A}^1 \hookrightarrow \mathbb{P}^1, t \mapsto [t, 1] \), to get a rational map \( \bar{\phi} : \mathbb{P}^2 \to \mathbb{P}^1 \). What is \( \text{dom}(\bar{\phi}) \)?

3. Cremona transformations Consider the rational map \( \mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^2 \)\
\[ [a_0, a_1, a_2] \mapsto [a_1 a_2, a_0 a_2, a_0 a_1] \]
(a) What is \( \text{dom}(\phi) \)?
(b) Show that \( \phi \) is birational, and is its own inverse.
(c) Find open subsets \( U, V \subset \mathbb{P}^2 \) such that \( \phi|_U : U \to V \) is an isomorphism.

4. Veronese embeddings Fix natural numbers \( m \) and \( d \). The set of all homogeneous polynomials of degree \( d \) in the variables \( X_0, \ldots, X_m \) is a vector space over \( k \) of dimension \( \binom{m+d}{d} \). Let
\[ E := \{ \xi = (e_0, \ldots, e_m) : \text{ each } e_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum e_i = d \} \]
\[ f_e := X^{e_0} \cdots X_m^{e_m} \]
Then \( \{ f_e : e \in E \} \) is a basis for the space of homogeneous polynomials of degree \( d \).
Let \( N = \binom{m+d}{d} - 1 \), and let \( \{ Z_e : e \in E \} \) be coordinates on \( \mathbb{P}^N \). Consider the rational map
\[ \mathbb{P}^m \xrightarrow{\nu} \mathbb{P}^N \]
\[ [a_0, \ldots, a_n] \mapsto [\cdots, f_e(a_0, \ldots, a_n), \cdots] \]
(a) Show that \( \nu \) is actually a morphism \( \mathbb{P}^m \to \mathbb{P}^N \).
(b) Show that \( \nu \) is an inclusion.

5. Continuation of 4 Let \( I \) be the ideal generated by the quadrics
\[ \{ Z_{e^{(1)}} Z_{e^{(2)}} - Z_{e^{(3)}} Z_{e^{(4)}} : (e^{(i)}) \in E, e^{(1)} + e^{(2)} = e^{(3)} + e^{(4)} \in \mathbb{Z}^{m+1} \} \]
Show that \( \nu(\mathbb{P}^m) = Z_\mathbb{P}(I) \), and that the inverse of \( \nu \) is a morphism from \( Z_\mathbb{P}(I) \) to \( \nu(\mathbb{P}^m) \).