Homework 4

Due: Wednesday, September 24

1. Let S be a graded ring, and let I and J be homogeneous ideals in S. Prove that each of the following is a homogeneous ideal .

   (a) $I + J$.

   (b) $I \cap J$.

   (c) IJ.

   (d) $\sqrt{I}$.

   You may, of course, assume that each of these is actually an ideal...

2. Let S be a graded ring, and let $I \subset S$ be a homogeneous ideal. Prove that I is prime if and only if for every pair of *homogeneous* $f, g \in S$ with $fg \in I$, one of f and g is in I. (HINT: *Write* $f = \sum_{d=m}^{M} f_d$*,* $g = \sum_{e=n}^{N} g_e$*, and consider the homogeneous pieces of the product* fg.)

3. Let $F = X_0^2 + X_1^2 - X_2^2$, and let $C = \mathcal{Z}_{\mathbb{P}}(F)$. Describe $C \cap U_i$ and $C \cap H_i$ for each coordinate $i = 0, 1, 2$.

Turn the page for a proof of Chow's theorem.

4. This problem is only valid over $\mathbb{C}$, and compares analytic objects to algebraic objects. Let π be the natural projection

$$\mathbb{C}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{P}^n_{\mathbb{C}}$$

Say that a function f on $\mathbb{C}^{n+1}$ is analytic in a neighborhood of the origin if there is a convergent power series

$$\sum_{\underline{e}=e_0,\cdots,e_n:e_j\in\mathbb{Z}_{\geq 0}} a_{\underline{e}} X_0^{e_0}\cdots X_n^{e_n}.$$

which agrees with f on some (analytic) neighborhood of 0.

Suppose $X \subset \mathbb{P}^n_{\mathbb{C}}$; let $Z = \mathcal{C}(X) = \pi^{-1}(X) \cup \{0\}$ be the affine cone over X.

Suppose f is analytic in some neighborhood of the origin. Write

$$f(z) = \sum_{d\geq 0} f_d(z)$$
$$f_d(z) = \sum_{\underline{e}:\sum e_i = d} a_{\underline{e}} z_0^{e_0}\cdots z_n^{e_n}.$$

Prove that if f vanishes on Z (in some neighborhood of the origin), then each f_d vanishes on Z.

If you like, you may proceed in the following way.

Define the function $g(z,t) = f(tz)$; here, $z \in \mathbb{A}^{n+1}_{\mathbb{C}}$, and $t \in \mathbb{C}$.

(a) Show $g(z,t)$ vanishes on Z for (sufficiently small) t.

(b) For a fixed value of z, consider the analytic function $g_z(t) = g(z,t) = f(tz)$. Show that the s^{th} derivative of $g_z(t)$ is

$$\frac{d^s}{dt^s} g_z(t) = \sum_{d\geq s} \frac{d!}{(d-s)!} f_d(z) t^{d-s}.$$

(c) Show that $f_d(z)$ vanishes on Z. (HINT: *Set $t = 0$, and take a Taylor series expansion of $g_z(t)$ centered at $t = 0$.*)

5. *Continuation of 4* Prove **Chow's Theorem:** Suppose $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a closed analytic space, in the sense that there is a collection of functions $\{g_\alpha\}$ on X such that $f_\alpha := g_\alpha \circ \pi$ is analytic on $\mathbb{A}^{n+1}$, and X is the vanishing locus of the g_α's. Show that X is algebraic, in the sense that it is the vanishing locus of a (finite) collection of homogeneous polynomials. *It suffices to show that $\mathcal{C}(X)$ is the vanishing locus of polynomials.*