Homework 3

Due: Wednesday, September 17

1. Let $H = \mathcal{Z}(x_1x_2 - 1) \subset \mathbb{A}^2$, and let $\mathbb{A}^1$ be the affine line with coordinate t, so that $k[\mathbb{A}^1] = k[t]$. Consider the morphism

$$\begin{array}{ccc} H & \xrightarrow{\phi} & \mathbb{A}^1 \\ (a_1, a_2) & \longmapsto & a_1 \end{array}$$

   (a) Describe the map $\phi^* : k[\mathbb{A}^1] \to k[H]$.

   (b) Is ϕ^* injective? Surjective?

   (c) Describe the image of ϕ. Is it a closed subset of $\mathbb{A}^1$?

2. Consider the morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{\beta} & \mathbb{A}^2 \\ (a_1, a_2) & \longmapsto & (a_1, a_1a_2) \end{array}$$

   (a) Is β injective? Explain.

   (b) Is β surjective? Explain.

   (c) Describe open subsets $U \subset \mathbb{A}^2$ and $V \subset \mathbb{A}^2$ such that β gives an isomorphism $\beta|_U : U \to V$.

3. (a) Let $W \subset \mathbb{A}^n$ be closed, and suppose that $U \subset W$. Show that $\mathcal{I}(U) = \mathcal{I}(W)$ if and only if U is dense in W.

   (b) Let $\phi : V \to W$ be a morphism. Show that the image of ϕ is dense in W if and only if $\phi^* : k[W] \to k[V]$ is injective. *Such a morphism is called* dominant.

4. (a) Let

$$\Delta_{\mathbb{A}^n} = \{(P, P) : P \in \mathbb{A}^n\} \subset \mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}.$$

   Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be coordinates on $\mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$. What is $\mathcal{I}(\Delta_{\mathbb{A}^n})$? Show that $\Delta_{\mathbb{A}^n}$ is closed.

   (b) Suppose $V \subset \mathbb{A}^n$ is closed. Show that

$$\Delta_V := \{(P, P); P \in V\} \subset \mathbb{A}^n \times \mathbb{A}^n$$

   is closed. (HINT: $V \times V \subset \mathbb{A}^{2n}$ *is closed.*)

5. Let $\phi : V \to W$ be a morphism.

   (a) The graph of ϕ is
   $$\Gamma_\phi := \{(P, \phi(P)) : P \in V\} \subset V \times W.$$
   Show that Γ_ϕ is closed. (HINT: *Consider the inverse image of* Δ_W *under* $(\phi \times \mathrm{id}_W) : V \times W \to W \times W$.)

   (b) Let $\psi : V \to W$ be a morphism. Show that $\{x \in V : \phi(x) = \psi(x)\}$ is closed.