Homework 2
Due: September 10

1. (a) Find polynomials
$$a(x) = \sum_{j=0}^{4} a_i x^i \text{ and } b(x) = \sum_{j=0}^{4} b_i x^i$$
such that
$$a(x)\cdot(x^2+1) + b(x)\cdot(x^3+1) = 1.$$
(HINT: *Solve for a_i and b_i.*)

   (b) Suppose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$ have no common zero. Suppose you know there is an N such that there are polynomials $g_1, \cdots, g_r \in k[x_1, \cdots, x_n]$ such that $\deg f_i g_i \le N$ and
$$\sum f_i g_i = 1.$$
Explain (briefly) how you would use linear algebra to find such polynomials.

   An *effective nullstellensatz* gives a computable value of N in terms of n, r, and the degree $f_1, \cdots, f_r$. See, e.g., J. Kollár, *Sharp effective Nullstellensatz*, JAMS 1 (1988), 963-765; and Z. Jelonek, On the effecitve Nullstellensatz, *Inv. Math.* 162 (2005), 1–17.

2. There is a natural identification (of sets) $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^2$. Show that the Zariski topology on $\mathbb{A}^2$ is strictly finer than the product topology of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

   Concretely, show:

   (a) Suppose $C_1, \cdots, C_r$ and $D_1, \cdots, D_r$ are closed subsets of $\mathbb{A}^1$. Then
$$\cup_{i=1}^{r} C_i \times D_i \subset \mathbb{A}^2 \tag{1}$$
is closed.

   (b) Find a set $S \subset \mathbb{A}^2$ which is closed but is *not* of the form (1).

3. If $f \in k[x_1, \cdots, x_n]$, the associated distinguished* affine open set is
$$D(f) := \{P \in \mathbb{A}^n : f(P) \neq 0\}.$$

   (a) Suppose $f, g \in k[x_1, \cdots, x_n]$. Show that $D(fg) = D(f) \cap D(g)$.

   (b) Show that the collection of distinguished open sets in $\mathbb{A}^n$ is a basis for the Zariski topology on $\mathbb{A}^n$.

   Recall that if X is a topological space, then a collection of open subsets $\mathcal{C}$ is a basis for the topology on X if for every open set U of X, and each $x \in U$, there is some $V \in \mathcal{C}$ such that
$$x \in V \subseteq U.$$

*or principal, or standard, or basic

4. (a) A topological space X is called quasicompact if every open cover admits a finite subcover.

   Suppose $\mathcal{C}$ is a basis for the topology of X. Prove that X is quasicompact if and only if every open covering $X = \cup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{C}$ admits a finite subcover.

   (b) Prove that $\mathbb{A}^n$ is quasicompact.