Homework 10

Due: Friday, Nov 21

The first two problems were used in our proof of Bézout's theorem.

1. The *dual space* of $\mathbb{P}^2$ is $\mathbb{P}^{2*}$, the space of lines in $\mathbb{P}^2$. In fact, $\mathbb{P}^{2*}$ is isomorphic to $\mathbb{P}^2$; a point $[a_0, a_1, a_2] \in \mathbb{P}^{2*}$ corresponds $\mathcal{Z}_{\mathbb{P}}(a_0X_0 + a_1X_1 + a_2X_2)$. (Note that is well-defined on equivalence classes!)

   Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous form, and let $X = \mathcal{Z}(F)$ be the associated plane curve, with smooth locus X^{sm}.

   Show that the map

$$\begin{array}{ccc} X^{\text{sm}} & \xrightarrow{\phi} & \mathbb{P}^{2*} \\ P & \longmapsto & T_PX \end{array}$$

   (where T_PX is the closure of the *external* tangent space to X at P) is a morphism, by giving an explicit formula for ϕ in terms of F and the coordinates on $\mathbb{P}^2$.

   The closure of the image is called the dual curve X^*.

2. Continue to assume $X = \mathcal{Z}(F) \subset \mathbb{P}^2$.

   (a) Show that the set of $L \in \mathbb{P}^{2*}$ which pass through a singular point of X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (b) Show that the set of $L \in \mathbb{P}^{2*}$ which are tangent to X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (c) Suppose $\deg F = d$. Show that there is an open subset $U \subset \mathbb{P}^{2*}$ such that for each $L \in U$, $L \cap X$ consists of exactly d points.

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3. Suppose $F \in k[X_0, X_1, X_2]$ is an irreducible homogeneous form of degree d. Give an upper bound for the number of singular points of the curve $\mathcal{Z}(F)$. (HINT: *Bézout!*)

4. (a) Suppose $g(z) \in k[z]$ is a polynomial. Show (directly) that $(z - \alpha)$ is a multiple root of $g(z)$ if and only if $(z-\alpha)|g(z)$ and $(z-\alpha)|g'(z)$. Show that $g(z)$ has repeated roots if and only if $\gcd(g(z), g'(z)) \neq 1$.

   (b) Suppose Y is a normal affine variety, and that X is a normal variety equipped with a finite map $\phi : X \to Y$ such that $k[X] \cong k[Y]/(f(z))$, where $f(z) = z^d + \sum_{i=0}^{d-1} a_i z^i$ with $a_i \in k[Y]$.

   Suppose $Q \in Y$, and let $f_Q(z) = z^d + \sum_i a_i(Q) z^i \in k[z]$. Show that ϕ is smooth on $\phi^{-1}(Q)$ if and only if $\gcd(f_Q(z), f'_Q(z)) = 1$.

5. Inside $\mathbb{P}^n$, consider the Fermat hypersurface of degree d,

$$H_{d,n} := \mathcal{Z}(X_1^d + \cdots + X_n^d - X_0^d).$$

Let $C \subset \mathbb{P}^n$ be a curve of degree e. Prove that $C \cap H_{d,n}$ consists of de points (when counted with multiplicity). *Assume that d is invertible in k, and that C doesn't pass through the point $[1, 1, 0, 0, \cdots, 0]$. If you need to place other restrictions on C, that's okay, too.*

6. Fix some $\lambda \in k$ such that the polynomial

$$f(x, y) = xy^2 + x^2y + x + \lambda y \in k[x, y]$$

is irreducible,* and let $X = \mathcal{Z}(f) \subset \mathbb{A}^2$.

Consider the map

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & \mathbb{A}^1 \\ (a, b) & \longmapsto & a \end{array}$$

(a) Show that the map ϕ is surjective and quasifinite. What is its (generic) degree?

(b) Show that ϕ is *not* finite. (HINT: *What is the fiber $\phi^{-1}(0)$, counted with appropriate multiplicity?*)

7. A weak version of the Cohen-Seidenberg going-up theorem says:

Theorem Let $B \subseteq A$ be rings, A integral over B, and let $\mathfrak{q}$ be a prime ideal of B. Then there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap B = \mathfrak{q}$.

(a) Show that a finite dominant map between irreducible quasiprojective varieties is surjective.

(b) Give an example of a dominant quasifinite map which is not surjective.

*Don't get distracted by λ; it's just chosen so that X is irreducible. If you like, assume $k = \mathbb{C}$ and $\lambda = 2$ and proceed.