Homework 1

Due: Wednesday, September 3

1. Let $S \subset \mathbb{A}^n$ be any subset. Prove that

$$\mathcal{I}(S) := \{f \in k[x_1, \cdots, x_n] : \forall P \in S, f(P) = 0\}$$

is an ideal of $k[x_1, \cdots, x_n]$.

2. (a) Define affine sets by

$$X = \{(a_1, a_2) : a_2^2 = a_1^3 - a_1\} \subset \mathbb{A}^2$$
$$Y = \{(b_1, b_2, b_3) : b_2^2 + b_1 = b_3 b_1 \text{ and } b_1^2 = b_3\} \subset \mathbb{A}^3$$

Consider the map

$$\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^3$$

$$(a_1, a_2) \longmapsto (a_1, a_2, a_1^2)$$

Show that $\alpha(X) \subseteq Y$.

(b) Define ideals by

$$I = (x_2^2 - x_1^3 + x_1) \subset k[x_1, x_2]$$
$$J = (t_2^2 + t_1 - t_3 t_1, t_1^2 - t_3) \subset k[t_1, t_2, t_3]$$

Consider the map

$$k[t_1, t_2, t_3] \xrightarrow{\beta} k[x_1, x_2]$$

$$t_1 \longmapsto x_1$$

$$t_2 \longmapsto x_2$$

$$t_3 \longmapsto x_1^2$$

Show that $\beta(J) \subseteq I$.

3. Let $\phi : R \to S$ be a ring homomorphism, and let $J \subset S$ be an ideal. Let $I = \phi^{-1}(J)$.

(a) Show that I is an ideal of R.

(b) Show that if J is prime, then I is prime.

(c) Give an example to show that even if J is maximal, I need not be maximal.

Extra: Suppose that $R = k[x_1, \cdots, x_m]/\mathfrak{a}$ and $S = k[y_1, \cdots, y_n]/\mathfrak{b}$, where k is algebraically closed. Show that if $J \subset S$ is maximal, then $\phi^{-1}(J)$ is maximal.

4. Let k be an algebraically closed field, and suppose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$. Show that there is no common solution $f_1 = f_2 = \cdots = 0$ if and only if there are $a_1, \cdots, a_r \in k[x_1, \cdots, x_n]$ such that

$$\sum_{i=1}^{r} a_i f_i = 1.$$