Homework 9

Due: Friday, October 27

1. The version of Noether's normalisation lemma we stated in class is quite geometric:

   Theorem *Suppose* $X \subset \mathbb{P}^n$ *is an irreducible subvariety,* $\dim X = r$*. Then there is a linear subspace* $E \subset \mathbb{P}^n$ *of dimension* $n - r - 1$ *such that* $E \cap X = \emptyset$*. Fix any such* E*. The projection* π_E *yields a finite-to-one map*

   $$X \xrightarrow{\pi_E} \mathbb{P}^r$$

   The homogeneous coordinate ring $k_{\text{homog}}[X] = k[X_0, \cdots, X_n]/I(X)$ *is a finitely generated module over* $k[Y_0, \cdots, Y_r] = k_{\text{homog}}[E]$*.*

   Explain how to deduce the version usually stated in commutative algebra books:

   Let R *be a finitely generated algebra over an algebraically closed field* k*. Then there exist elements* $y_1, \cdots, y_r \in R$*, algebraically independent over* k*, such that* R *is finite as a* $k[y_1, \cdots, y_r]$*-module.*

   (Actually, this is true for an arbitrary, not-necessarily-algebraically-closed base field k.)

2. [*] If $f \in k[x_1, \cdots, x_n]$, and $P = (a_1, \cdots, a_n) \in \mathbb{A}^n$, the *linearization of* f *at* P *is*

   $$d_P(f) = f(P) + \sum_{i=1}^{n} (\frac{\partial}{\partial x_i} f)(P)(x_i - a_i).$$

   Suppose that $I = (f_1, \cdots, f_r) \subset k[x_1, \cdots, x_n]$ and that $P \in \mathcal{Z}(I)$. Let $d_P(I)$ be the ideal generated by the linearizations of all elements of I:

   $$d_P(I) = (\{d_P(f) : f \in I\})k[x_1, \cdots, x_n].$$

   Show that $d_P(I) = (d_P(f_1), \cdots, d_P(f_r))$. (HINT: *If* $f \in I$ *and* $g \in k[x_1, \cdots, x_n]$*, what is* $d_P(fg)$*?*)

3. [*] Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. We say X is singular at P if $\dim T_P X > \dim(X)$, and is smooth or nonsingular at P if $\dim T_P X = \dim(X)$. (We will prove that for every point P, $\dim T_P X \geq \dim(X)$.)

   Suppose $X = \mathcal{Z}(f)$ is an irreducible hypersurface.

   (a) Suppose $P \in X$. Show that X is singular at P if and only if $(\frac{\partial}{\partial x_i} f)(P) = 0$ for each $i = 1, \cdots, n$.

   (b) Show that the set of smooth points of X is a non-empty open subset.

4. Let $f(x,y) \in k[x,y]$ be a non-constant polynomial, and let $X = \mathcal{Z}(f)$. If $P = (a,b) \in \mathbb{A}^2$, expand f in a Taylor series centered at P:

$$\begin{aligned} f &= f(P) + (\frac{\partial}{\partial x} f(P)(x-a) + \frac{\partial}{\partial y} f(P)(y-b)) + \\ &\quad \frac{1}{2}(\frac{\partial^2}{\partial x^2} f(P)(x-a)^2 + 2\frac{\partial^2}{\partial x \partial y} f(P)(x-a)(y-b) + \frac{\partial^2}{\partial y^2} f(P)(y-b)^2) + \cdots \\ &= \sum f_i \end{aligned}$$

where $f_i \in k[x,y]$ is homogeneous of degree i.

The multiplicity of X at P, $\mu_P(X)$, is the smallest r such that $f_r \neq 0$.

(a) Show that $P \in X \iff \mu_P(X) \geq 1$.

(b) Show that X is singular at $P \iff \mu_P(X) \geq 2$.

5. Let $f = y^4 - 2y^3 + y^2 - 3x^2y + 2x^4$, and let $X = \mathcal{Z}(f) \subset \mathbb{A}^2$.

(a) Find all singular points of X.

(b) For each singular point P, compute $\mu_P(X)$.