Homework 8

Due: Friday, October 20

Henceforth, problems with asterisks are candidates for in-class presentations.

1. Let $Z_0, \cdots, Z_3$ be coordinates on $\mathbb{P}^3$. Consider the morphism

$$\mathbb{P}^1 \xrightarrow{\nu} \mathbb{P}^3$$

$$[a, b] \longmapsto [a^3, a^2b, ab^2, b^3]$$

and the quadrics

$$F_0 = Z_0Z_2 - Z_1^2$$
$$F_1 = Z_0Z_3 - Z_1Z_2$$
$$F_2 = Z_1Z_3 - Z_2^2$$

Let $C = \nu(\mathbb{P}^1)$.

(a) Convince yourself that $C = \mathcal{Z}_{\mathbb{P}}(F_0, F_1, F_2)$. *You need not hand this in.*

(b) Show that for each $0 \le i < j \le 2$, $\mathcal{Z}_{\mathbb{P}}(F_i, F_j) \supsetneq C$. (HINT: *In fact,* $\mathcal{Z}_{\mathbb{P}}(F_i, F_j) = C \cup L$, *where L is some line of the form* $\mathcal{Z}(Z_q, Z_r)$.)

Note that $\dim \mathcal{Z}(F_0, F_1) = \dim \mathcal{Z}(F_0, F_1, F_2)$; *imposing an extra condition need not force the dimension to drop!*

2. [*] A *ring variety* is a variety X equipped with

addition A morphism $\alpha : X \times X \to X$;

multiplication A morphism $\mu : X \times X \to X$;

additive inverse A morphism $\iota : X \to X$;

additive identity An element $\zeta \in X$

such that (X, α, ζ) is a group variety, and multiplication satisfies the obvious axioms.

Suppose that X is a projective irreducible ring variety. Prove that the multiplication map must be trivial.

You may, and should, use basic properties of rings. This answers a question somebody asked me in Laramie.

3. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. Suppose there exists some $d \in \mathbb{N}$ such that for every pair of homogeneous forms $F, G \in k[X_0, \cdots, X_n]$ with $\deg F = \deg G = d$ and $G \notin \mathcal{I}_{\mathbb{P}}(X)$, the rational function $P \mapsto F(P)/G(P)$ is constant. Show that X is a point.

This doesn't use anything recent from class, but it helps a lot on the next problem.

4. * Suppose $X \subset \mathbb{P}^n$ is a projective irreducible variety. Let H be homogeneous of degree d, and let Y be the hypersurface $Y = \mathcal{Z}_{\mathbb{P}}(H)$.

   (a) Suppose $X \cap Y = \emptyset$. Let F be any homogeneous form of degree d. Show that the function on X given by $P \mapsto F(P)/H(P)$ is constant.

   (b) Continue to suppose $X \cap Y = \emptyset$. Show that for any homogeneous polynomials F and G of degree d, where $G \notin \mathcal{I}_{\mathbb{P}}(X)$, the rational function on X $P \mapsto F(P)/G(P)$ is constant.

   (c) Suppose $X \subset \mathbb{P}^n$ is a projective variety of positive dimension. Let $Y \subset \mathbb{P}^n$ be a hypersurface. Show that $X \cap Y$ is nonempty.