Homework 7
Due: Monday, October 9

1. Let $P = [0, \cdots, 0, 1] \in \mathbb{P}^n$. Consider the rational map

$$\mathbb{P}^n \overset{\pi_P}{\dashrightarrow} \mathbb{P}^{n-1}$$

$$[a_0, \cdots, a_n] \longmapsto [a_0, \cdots, a_{n-1}],$$

the *projection from P*.

   (a) What is $\operatorname{dom}(\pi_P)$?
   (b) Show that π is surjective.
   (c) If $Q \in \mathbb{P}^n$, what is $\pi_P^{-1}(\pi(Q))$? Describe this set geometrically.

2. Let V and W be k-vector spaces of dimensions m and n, respectively. After choosing a basis on V and W, we may identify $\mathbb{A}^{mn}$ with (the set of $m \times n$ matrices with entries in k, and thus with) $\operatorname{LinMap}(V, W)$.

   (a) Suppose $\dim V = \dim W$. Prove that the set of elements of $\operatorname{LinMap}(V, W)$ which are actually isomorphisms is a Zariski open subset of $\operatorname{LinMap}(V, W)$.
   (b) Let r be a nonnegative integer. Show that the set

   $$M_r := \{\alpha \in \operatorname{LinMap}(V, W) : \dim(\alpha(V)) \leq r\}$$

   is a Zariski closed subset of $\operatorname{LinMap}(V, W)$. (Here, *dim* means dimension as vector space.)

3. Recall that $\dim(X)$ is the transcendence degree of $k(X)$ over k.

   (a) Suppose X and Y are irreducible quasiprojective varieties, and that there exists a dominant rational map $X \dashrightarrow Y$. Prove that $\dim(X) \geq \dim(Y)$.
   (b) Let X be an irreducible affine variety, and let $Y \subseteq X$ be an irreducible closed subvariety. Show that $\dim(Y) \leq \dim X$.
   In fact, much more is true; $\dim Y = \dim X$ *if and only if* $Y = X$.

4. If X is a topological space, the topological dimension of X, $\operatorname{tdim}(X)$, is the supremum of the lengths of all chains

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n,$$

where each Z_i is a closed, irreducible subset of X.

If R is a ring, the height of a prime ideal $\mathfrak{p} \subset R$ is the supremum of the lengths of all chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \mathfrak{p}_n = \mathfrak{p}$$

of distinct prime ideals. The Krull dimension of R, $\operatorname{kdim} R$, is the supremum of the heights of all prime ideals.

Let Y be an irreducible affine variety.

(a) Prove that $\operatorname{tdim}(Y) = \operatorname{kdim}(k[Y])$.

(b) Use the following result from commutative algebra to show that $\operatorname{tdim}(Y) = \dim(Y)$.

Theorem Let R be an integral domain which is finitely generated as a k-algebra. Then $\operatorname{kdim} R = \operatorname{tr.deg.}(\operatorname{Frac}(R)/k)$.