1. Prove that a rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is actually regular.

2. (a) Let $\phi$ be the rational function on $\mathbb{P}^2$ given by $\phi = X_1/X_0$. What is $\text{dom}(\phi)$? Describe the corresponding regular function on $\text{dom}(\phi)$.

   (b) Now, think of this function as a rational map from $\mathbb{P}^2$. Compose this with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $t \mapsto [t, 1]$, to get a rational map $\bar{\phi} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. What is $\text{dom}(\bar{\phi})$?

3. Cremona transformations Consider the rational map

   $\mathbb{P}^2 \stackrel{\phi}{\dashrightarrow} \mathbb{P}^2$

   $[a_0, a_1, a_2] \mapsto [a_1a_2, a_0a_2, a_0a_1]$

   (a) What is $\text{dom}(\phi)$?

   (b) Show that $\phi$ is birational, and is its own inverse.

   (c) What is $\text{dom}(\phi^{-1})$?

4. Veronese embeddings Fix natural numbers $m$ and $d$. The set of all homogeneous polynomials of degree $d$ in the variables $X_0, \cdots, X_m$ is a vector space over $k$ of dimension $\binom{m+d}{d}$. Let

   $E := \{ \mathbf{e} = (e_0, \cdots, e_m) : \text{each } e_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum e_i = d \}$

   $f_\mathbf{e} := X_0^{e_0} \cdots X_m^{e_m}$

   Then $\{ f_\mathbf{e} : \mathbf{e} \in E \}$ is a basis for the space of homogeneous polynomials of degree $d$.

   Let $N = \binom{m+d}{d} - 1$, and use these as coordinates on $\mathbb{P}^N$. Consider the rational map

   $\mathbb{P}^m \dashrightarrow \mathbb{P}^N$

   $[a_0, \cdots, a_n] \mapsto [\cdots, f_\mathbf{e}(a_0, \cdots, a_n), \cdots]$  

   (a) Show that $\nu$ is actually a morphism $\mathbb{P}^m \rightarrow \mathbb{P}^N$.

   (b) Show that $\nu$ is an inclusion.

5. Continuation of 4 Let $I$ be the ideal generated by the quadrics

   $\{ f_\mathbf{e}^{(1)} f_\mathbf{e}^{(2)} - f_\mathbf{e}^{(3)} f_\mathbf{e}^{(4)} : \mathbf{e}^{(i)} \in E, \mathbf{e}^{(1)} + \mathbf{e}^{(2)} = \mathbf{e}^{(3)} + \mathbf{e}^{(4)} \in \mathbb{Z}^{m+1} \}$

   Show that $\nu(\mathbb{P}^m) = \mathbb{Z}_\nu(I)$, and that the inverse of $\nu$ is a morphism from $\mathbb{Z}_\nu(I)$ to $\nu(\mathbb{P}^m)$.