Homework 5

Due: Friday, September 22

1. Let $F = A_0X_0 + A_1X_1 + A_2X_2$ and let $G = B_0X_0 + B_1X_1 + B_2X_2$, where at least one A_i and at least one B_j are nonzero. Consider the lines in the projective plane $L = \mathcal{Z}_{\mathbb{P}}(F) \subset \mathbb{P}^2$ and $M = \mathcal{Z}_{\mathbb{P}}(G) \subset \mathbb{P}^2$.

   (a) Show that $L = M$ if and only if there exists some $\lambda \in k^\times$ such that, for each i, $B_i = \lambda A_i$.

   (b) Suppose $L \neq M$. Prove that the intersection $L \cap M$ consists of a unique point, P.

   (c) Suppose $L \neq M$. Give a criterion for when $L \cap M \in H_0$.

2. A hypersurface defined by a linear polynomial is called a hyperplane.

   (a) Suppose $Y \subset \mathbb{P}^n$ is a projective variety. Show that the following conditions are equivalent:

      i. $\mathcal{I}_{\mathbb{P}}(Y)$ can be generated by a set of linear polynomials.

      ii. Y can be written as an intersection of hyperplanes.

      Such a variety is called a linear variety.

   (b) Consider $\mathbb{A}^{n+1}$ as an $n+1$-dimensional vector space over k. Show that the affine cone over a linear variety is a sub-vector space of $\mathbb{A}^{n+1}$.

3. Let $F = X_0^2 + X_1^2 - X_2^2$, and let $C = \mathcal{Z}_{\mathbb{P}}(F)$. Describe $C \cap U_i$ and $C \cap H_i$ for each coordinate $i = 0, 1, 2$.

4. Consider the map

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\quad\phi\quad} \mathbb{P}^3$$

$$(a_0 : a_1) \times (b_0 : b_1) \longmapsto (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1)$$

   Show that the image of ϕ is an algebraic set. (HINT: *There exists a form* $F \in k[X_0, \cdots, X_3]$, *homogeneous of degree two, such that* $\operatorname{im}(\phi) = \mathcal{Z}_{\mathbb{P}}(F)$.)

5. Prove that any regular function on $\mathbb{P}^n$ is constant. (HINT: *Mimic our proof from class for* $\mathbb{P}^1$. *In fact, you can show that if f is a rational function on* $\mathbb{P}^n$, *and if f is regular on distinct affine patches U_i and U_j, then f is constant.*)